Let U and V be vector spaces over a field F and $\dim U = n$. Let $T: U \to V$ be a linear operator, then $\text{rank}\,(T) + \text{nullity}(T) = \cdots$

a. 0
b. 1
c. $n - 1$
d. n

Solution.

Use the rank-nullity theorem to prove it. The theorem states that the rank and the nullity of a matrix add up to the number of columns of the matrix. Specifically, if A is an m-by-n matrix (with m rows and n columns) over some field, then

$$\text{rank}\,A + \text{nullity}\,A = n.$$

So we have two vector spaces U and V over a field F and $T: U \to V$. Then the rank of T is the dimension of the image of T and the nullity of T is the dimension of the kernel of T, so we have

$$\dim(\text{im}\,T) + \dim(\ker T) = \dim U$$

or, equivalently,

$$\text{rank}\,T + \text{nullity}\,T = \dim U$$

Such that $\dim U = n$ we have

$$\text{rank}\,T + \text{nullity}\,T = n$$

Answer:

$d.$ $\text{rank}\,T + \text{nullity}\,T = n$