Task. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined by $T(a,b,c) = (a-b, b-c, c-a)$. Find the matrix with respect to basis (v_1, v_2, v_3), where

$$v_1 = (1,0,1), \qquad v_2 = (0,1,1), \qquad v_3 = (1,1,0).$$

Is this matrix invertible? If so find inverse corresponding to formula for T^{-1}.

Solution. The matrix of T has the following form:

$$T = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

The transition matrix to the basis (v_1, v_2, v_3) has the form

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Therefore the matrix of T in the basis (v_1, v_2, v_3) is

$$ATA^{-1}.$$

Let us compute the inverse of A.

First find the determinant of A:

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \cdot 1 \cdot 0 + 0 \cdot 1 \cdot 1 + 1 \cdot 0 \cdot 1 - 1 \cdot 1 \cdot 1 - 1 \cdot 1 \cdot 1 - 0 \cdot 0 \cdot 0 = -2.$$

Now compute the minores

$$M_{11} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot 0 - 1 \cdot 1 = -1$$

$$M_{12} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1$$

$$M_{13} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 \cdot 1 - 1 \cdot 1 = -1$$

$$M_{21} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 0 \cdot 0 - 1 \cdot 1 = -1$$

$$M_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \cdot 0 - 1 \cdot 1 = -1$$

$$M_{23} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 1 = 1$$

$$M_{31} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = 0 \cdot 1 - 1 \cdot 1 = -1$$

$$M_{32} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 1 \cdot 0 = 1$$

$$M_{33} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \cdot 1 - 0 \cdot 0 = 1$$

Then the inverse of A is given by the formula:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} =$$

Now compute

$$AT = \begin{pmatrix}1&0&1\\0&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{pmatrix}$$

$$= \begin{pmatrix}1\cdot1+0\cdot0+1\cdot(-1) & 1\cdot(-1)\ +0\cdot1+1\cdot0 & 1\cdot0+0\cdot(-1)+1\cdot1\\ 0\cdot1+1\cdot0+1\cdot(-1) & 0\cdot(-1)\ +1\cdot1+1\cdot0 & 0\cdot0+1\cdot(-1)+1\cdot1\\ 1\cdot1+1\cdot0+0\cdot(-1) & 1\cdot(-1)\ +1\cdot1+0\cdot0 & 1\cdot0+1\cdot(-1)+0\cdot1\end{pmatrix}$$

$$= \begin{pmatrix}0&-1&1\\-1&1&0\\1&0&-1\end{pmatrix}$$

Hence

$$ATA^{-1} = \begin{pmatrix}-1&1&-1\\1&-1&-1\\-1&-1&1\end{pmatrix}\begin{pmatrix}0&-1&1\\-1&1&0\\1&0&-1\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}0\cdot(-1)-1\cdot1+1\cdot(-1) & 0\cdot1-1\cdot(-1)+1\cdot(-1) & 0\cdot(-1)-1\cdot(-1)+1\cdot1\\ -1\cdot(-1)+1\cdot1+0\cdot(-1) & -1\cdot1+1\cdot(-1)+0\cdot(-1) & -1\cdot(-1)+1\cdot(-1)+0\cdot1\\ 1\cdot(-1)+0\cdot1-1\cdot(-1) & 1\cdot1+0\cdot(-1)-1\cdot(-1) & 1\cdot(-1)+0\cdot(-1)-1\cdot1\end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix}-2&0&2\\2&-2&0\\0&2&-2\end{pmatrix} = \begin{pmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{pmatrix}$$

Compute the determinant of T:

$$|T| = \begin{vmatrix}1&0&1\\0&1&1\\1&1&0\end{vmatrix} = 1\cdot1\cdot0+0\cdot1\cdot1+1\cdot0\cdot1-1\cdot1\cdot1-1\cdot1\cdot1-0\cdot0\cdot0=0.$$

It zero, so the transformation T is not invertible.

Answer. T is not invertible, so its inverse does not exist.

The matrix of T in the basis (v_1, v_2, v_3) is

$$\begin{pmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{pmatrix}$$