$\sec x + \tan x = 2 + \sqrt{5}$. Then find $\sin x + \cos x$

Solution.

Let's solve the equation for x:

$$\sec x + \tan x = 2 + \sqrt{5}$$

Rewrite it in another form:

$$\frac{1}{\cos x} + \frac{\sin x}{\cos x} = 2 + \sqrt{5}$$

Use the **Weierstrass substitution**:

$$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} \; and \; \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

Substitute $t = \tan\frac{x}{2}$. Then

$$\sin x = \frac{2t}{1+t^2} \; and \; \cos x = \frac{1-t^2}{1+t^2}$$

Then our equation has the form:

$$\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} = 2 + \sqrt{5}$$

$$\frac{1+t^2+2t}{1-t^2} = 2 + \sqrt{5}$$

$$\frac{(1+t)^2}{(1-t)(1+t)} = 2 + \sqrt{5}$$

$$\frac{1+t}{1-t} = 2 + \sqrt{5}, \;\; t \neq -1$$

Solve the equation for t:

$$\frac{1+t}{1-t} - 2 - \sqrt{5} = 0$$

$$\frac{1+t-2-\sqrt{5}+2t+\sqrt{5}t}{1-t} = 0$$

Multiply by $1-t$:

$$-1-\sqrt{5}+3t+\sqrt{5}t = 0$$

$$\left(3+\sqrt{5}\right)t = 1+\sqrt{5}$$

$$t = \frac{1+\sqrt{5}}{3+\sqrt{5}}$$

Substitute back for $t = \tan\frac{x}{2}$:

$$\tan\frac{x}{2} = \frac{1+\sqrt{5}}{3+\sqrt{5}}$$

Take the inverse tangent of both sides:

$$\frac{x}{2} = \arctan\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right) + \pi k, \ \ k \in \mathbb{Z}$$

Then

$$x = 2\arctan\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right) + 2\pi k, \ \ k \in \mathbb{Z}$$

So find $\sin x + \cos x$:

$$\sin x + \cos x = \sin\left(2\arctan\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right) + 2\pi k\right) + \cos\left(2\arctan\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right) + 2\pi k\right) =$$

$$= \sin\left(2\arctan\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right)\right) + \cos\left(2\arctan\left(\frac{1+\sqrt{5}}{3+\sqrt{5}}\right)\right)$$

Let's calculate $\sin x$:

$$\frac{1+\sqrt{5}}{3+\sqrt{5}} = \frac{1+\sqrt{5}}{3+\sqrt{5}} \cdot \frac{3-\sqrt{5}}{3-\sqrt{5}} = \frac{3+2\sqrt{5}-5}{4} = \frac{\sqrt{5}-1}{2}$$

Use this formula $\sin 2x = 2\sin x \cos x$:

$$\sin\left(2\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right) = 2\sin\left[\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right]\cos\left[\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right]$$

Then use compositions of trig and inverse trig functions:

$$\sin(\arctan x) = \frac{x}{\sqrt{1+x^2}} \ \ and \ \ \cos(\arctan x) = \frac{1}{\sqrt{1+x^2}}$$

We have

$$2\sin\left[\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right]\cos\left[\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right] = 2 \cdot \frac{\left(\frac{\sqrt{5}-1}{2}\right)}{\sqrt{1+\left(\frac{3-\sqrt{5}}{2}\right)}} \cdot \frac{1}{\sqrt{1+\left(\frac{3-\sqrt{5}}{2}\right)}} =$$

$$= 2\frac{\left(\frac{\sqrt{5}-1}{2}\right)}{1+\left(\frac{3-\sqrt{5}}{2}\right)} = 2 \cdot \frac{\sqrt{5}-1}{5-\sqrt{5}} = 2 \cdot \frac{1}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

Similarly

Use the formula $\cos 2x = 1 - 2\sin^2 x$

$$\cos\left[2\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right] = 2\cos^2\left[\arctan\left(\frac{\sqrt{5}-1}{2}\right)\right] - 1 = \frac{2}{1+\left(\frac{3-\sqrt{5}}{2}\right)} - 1 = \frac{4}{2+3-\sqrt{5}} - 1 =$$

$$= \frac{4-5+\sqrt{5}}{5-\sqrt{5}} = \frac{1}{\sqrt{5}}$$

So

$$\sin x + \cos x = \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}} = \frac{3}{\sqrt{5}}$$

Answer:

$$\sin x + \cos x = \frac{3}{\sqrt{5}}$$