Task 1. The ordinate of a point which is $\sqrt{29}$ units from (5,1) is twice its abscissa. Find the point.

Solution. Let $A = (x, y)$ be the point which we are looking for, and $B = (5, 1)$. Then the ordinate y of A is twice its abscissa x, that is $y = 2x$, and so $A = (x, 2x)$.

Moreover, the distance $AB = \sqrt{29}$.

We should find x.

Let us write the distance AB:

$$AB = \sqrt{(x-5)^2 + (2x-1)^2} = \sqrt{29},$$

whence

$$(x-5)^2 + (2x-1)^2 = 29,$$

$$x^2 - 10x + 25 + 4x^2 - 4x + 1 = 29$$

$$5x^2 - 14x + 26 - 29 = 0$$

$$5x^2 - 14x - 3 = 0,$$

$$D = 14^2 + 4 * 5 * 3 = 256 = 16^2$$

$$x_1 = \frac{14+16}{2*5} = \frac{30}{10} = 3, \qquad x_2 = \frac{14-16}{2*5} = -\frac{2}{10} = -0.3.$$

Hence

$$y_1 = 2x_1 = 2 * 3 = 6, \qquad y_2 = 2x_2 = 2 * (-0.2) = -0.4.$$

Answer. There are two points

$$A_1(3, 6), \qquad A_2(-0.2, -0.4).$$

Task 2. A point $A(x, y)$ is at a distance $4\sqrt{2}$ from $B = (-3/2, -5/2)$ and at a distance $2\sqrt{5}$ from $C = (9/2, -5/2)$. Find the point.

Solution. Let us write the formulas for the distances AB ad AC:

$$AB = \sqrt{(x - (-3/2))^2 + (y - (-5/2))^2} = 4\sqrt{2}$$
$$AC = \sqrt{(x - 9/2)^2 + (y - (-5/2))^2} = 2\sqrt{5}$$

From the first equation we get

$$(x + 1.5)^2 + (y + 2.5)^2 = (4\sqrt{2})^2$$
$$x^2 + 3x + 2.25 + y^2 + 5y + 6.25 = 32$$
$$x^2 + 3x + y^2 + 5y + 8.5 = 32$$
$$x^2 + 3x + y^2 + 5y = 23.5.$$

The second equation gives

$$(x - 4.5)^2 + (y + 2.5)^2 = (2\sqrt{5})^2$$
$$x^2 - 9x + 20.25 + y^2 + 5y + 6.25 = 50$$
$$x^2 - 9x + y^2 + 5y + 26.5 = 50$$
$$x^2 - 9x + y^2 + 5y = 23.5.$$

Thus we get the following system

$$\begin{cases} x^2 + 3x + y^2 + 5y = 23.5 \\ x^2 - 9x + y^2 + 5y = 23.5 \end{cases}$$

Subtracting the second equation from the first we get

$$x^2 + 3x + y^2 + 5y - (x^2 - 9x + y^2 + 5y) = 23.5 - 23.5$$
$$12x = 0 \qquad \Rightarrow \qquad x = 0.$$

Sinstituting x into the first equation we obtain

$$y^2 + 5y = 23.5$$
$$y^2 + 5y - 23.5 = 0$$
$$D = 25 + 4 * 23.5 = 119$$
$$y_1 = \frac{-5 + \sqrt{119}}{2}, \qquad y_2 = \frac{-5 - \sqrt{119}}{2}.$$

Answer. There are two points:

$$A_1 = \left(0, \frac{-5 + \sqrt{119}}{2}\right), \qquad A_2 = \left(0, \frac{-5 - \sqrt{119}}{2}\right).$$

Task 3. Find the coordintes of a point $P = (x, y)$ equidistant from $A = (-2, 8)$, $B = (1, -1)$ and $C = (-8, -4)$.

Solution. Let us write squares of distances PA, PB and PC:

$$PA^2 = (x+2)^2 + (y-8)^2 = x^2 + 4x + 4 + y^2 - 16y + 64 = x^2 + 4x + y^2 - 16y + 68,$$
$$PB^2 = (x-1)^2 + (y+1)^2 = x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 - 2x + y^2 + 2y + 2,$$
$$PC^2 = (x+8)^2 + (y+4)^2 = x^2 + 16x + 64 + y^2 + 8y + 16 = x^2 + 16x + y^2 + 8y + 80.$$

We have that these distances coincides, so

$$PA^2 = PB^2, \qquad PA^2 = PC^2.$$

Let us write down $PA^2 = PB^2$:

$$x^2 + 4x + y^2 - 16y + 68 = x^2 - 2x + y^2 + 2y + 2,$$
$$4x - 16y + 68 = -2x + 2y + 2,$$
$$6x - 18y = -66,$$
$$x - 3y = -11.$$

Similarly, write down $PA^2 = PC^2$:

$$x^2 + 4x + y^2 - 16y + 68 = x^2 + 16x + y^2 + 8y + 80$$
$$4x - 16y + 68 = 16x + 8y + 80$$
$$-12x - 24y = 12$$
$$x + 2y = -1.$$

Thus we get the following system

$$\begin{cases} x - 3y = -11 \\ x + 2y = -1. \end{cases}$$

Subtracting the second equation from the first we get

$$x - 3y - (x + 2y) = -11 - (-1)$$
$$-5y = -10 \qquad \Rightarrow \qquad y = 2$$

Hence, from $x + 2y = -1$ we get

$$x = -1 - 2y = -1 - 2 * 2 = -5.$$

Answer. $P(-5, 2)$.