Task. Let X be a set and $f : X \to \mathbb{R}$ be a function. Prove the following equality:

$$\sup(f) = -\inf(-f).$$

Solution. By definition $\sup(f)$ is a number such that $f(x) \leq \sup(f)$ for all $x \in X$, and for every $\varepsilon > 0$ there exists $x \in X$ such that

$$\sup(f) < f(x) + \varepsilon.$$

Similarly, $\inf(f)$ is the number such that $\sup(f) \leq f(x)$ for every $x \in X$, and for every $\varepsilon > 0$ there exists $x \in X$ such that

$$f(x) - \varepsilon < \inf(f).$$

In particular, if we know that

$$A \leq f(x) \leq B$$

for all $x \in A$, then

$$A \leq \inf(f) \leq \sup(f) \leq B.$$

Now we can deduce from latter property that

$$\sup(f) = -\inf(-f).$$

First we show that

$$\sup(f) \leq -\inf(-f).$$

We have that $f(x) \leq \sup(f)$ for all $x \in X$, which is equivalent to

$$-f(x) \geq -\sup(f)$$

whence

$$-f(x) \geq \inf(-f) \geq -\sup(f)$$

and so

$$-\inf(-f) \leq \sup(f). \tag{1}$$

Conversely, we have that $\inf(-f) \leq -f(x)$ for all $x \in X$, which is equivalent to

$$-\inf(-f) \geq f(x)$$

whence

$$-\inf(-f) \geq \sup(f) \geq f(x)$$

and so

$$-\inf(-f) \geq \sup(f). \tag{2}$$

Therefore from (1) and (2) we obtain that

$$\sup(f) = -\inf(-f).$$