Task. Find domain and range of the function

$$f(x) = \log_2 \log_{1/3} \sqrt{4-x^2}$$

Solution. First we find the domain of f. Notice that the domain for the function $g(x) = \sqrt{x}$ is determined by the inequality $x \geq 0$, while the domain of $h(x) = \log_a x$ is determined by the inequality $x > 0$.

Hence the domain of f consists of the solution of the following system:

$$\begin{cases} 4 - x^2 \geq 0 \\ \sqrt{4-x^2} > 0 \\ \log_{1/3} \sqrt{4-x^2} > 0 \end{cases}$$

Let us solve this system. The first two inequalities are equivalent to $4 - x^2 > 0$, and the third one is equivalent to $0 < \sqrt{4-x^2} < 1$ So we get the following system

$$\begin{cases} 0 < 4 - x^2 \\ 0 < \sqrt{4-x^2} < 1 \end{cases}$$

which reduces to the following

$$0 < 4 - x^2 < 1$$
$$-4 < -x^2 < -4 + 1$$
$$-4 < -x^2 < -3$$
$$3 < x^2 < 4$$

Hence the domain of the function is

$$D(f): \ (-2, -\sqrt{3}) \cup (\sqrt{3}, 2).$$

Let us compute the range of f, that is the subset $f(D(f))$ of $\mathbb{R}$.

Denote $a(x) = \sqrt{4-x^2}$. As noted above the domain $D(f)$ is determined by the inequality $0 < 4 - x^2 < 1$, whence

$$a(D(f))$$

coincides with the image of $(0, 1)$ under the function $\sqrt{x}$. Since $\sqrt{x}$ is monotone, $\sqrt{0} = 0$ and $\sqrt{1} = 1$, we obtain that

$$a(D(f)) = (0, 1).$$

Furhtermore the function $\log_{1/3}$ maps interval $(0, 1)$ onto the interval $(0, +\infty)$.

Finally, the function $\log_2$ maps $(0, +\infty)$ onto the interval $(-\infty, +\infty)$.

Hence the range $E(f)$ of f is $(-\infty, +\infty)$.

Answer. $D(f) = (-2, -\sqrt{3}) \cup (\sqrt{3}, 2), \qquad E(f) = (-\infty, +\infty).$