Task. Find the partial derivatives of

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}.$$

Solution. We will use the following two formulas:

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

$$\sqrt{x}' = (x^{1/2})' = \frac{1}{2}\,x^{\frac{1}{2}-1} = \frac{1}{2}\,x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

$$f'_x(x,y) = \left(\frac{xy}{\sqrt{x^2+y^2}}\right)'_x = \frac{(xy)'_x\sqrt{x^2+y^2} - xy(\sqrt{x^2+y^2})'_x}{\sqrt{x^2+y^2}^{\;2}}$$

$$= \frac{y\sqrt{x^2+y^2} - xy\,\frac{(x^2+y^2)'_x}{2\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y\sqrt{x^2+y^2} - xy\,\frac{2x}{2\sqrt{x^2+y^2}}}{x^2+y^2}$$

$$= \frac{y(x^2+y^2) - x^2y}{(x^2+y^2)\sqrt{x^2+y^2}} = \frac{yx^2+y^3-x^2y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}$$

Since $f(x,y) = f(y,x)$, we have that

$$f'_y(x,y) = f'_x(y,x) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}.$$

Answer.

$$f'_x(x,y) = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}}, \qquad f'_y(x,y) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}.$$