Prove the identity $\cos^3 2A + 3\cos 2A = 4(\cos^6 A - \sin^6 A)$

Solution.

Lets start with the right side. Use the power-reduction formula:

$$\cos^6 A = \left(\frac{1+\cos 2A}{2}\right)^3 = \frac{1}{8}(1 + 3\cos 2A + 3\cos^2 2A + \cos^3 2A)$$

$$\sin^6 A = \left(\frac{1-\cos 2A}{2}\right)^3 = \frac{1}{8}(1 - 3\cos 2A + 3\cos^2 2A - \cos^3 2A)$$

Then we add these expressions:

$4(\cos^6 A - \sin^6 A) = \frac{1}{2}(1 + 3\cos 2A + 3\cos^2 2A + \cos^3 2A - 1 + 3\cos 2A - 3\cos^2 2A + \cos^3 2A)$

And finally simplify the expressions:

$\frac{1}{2}(\cancel{1} + 3\cos 2A + 3\cancel{\cos^2 2A} + \cos^3 2A - \cancel{1} + 3\cos 2A - \cancel{3\cos^2 2A} + \cos^3 2A) =$
$\frac{1}{2}(6\cos 2A + 2\cos^3 2A) = 3\cos 2A + \cos^3 2A$

Check the left and right sides:

$$\cos^3 2A + 3\cos 2A = \cos^3 2A + 3\cos 2A$$

We have the identity.