Solve. Tangent of an angle using (m+n)cotQ=mcot A -ncot B
where m and n = a (each)
B =α +2θ
A = β
Q=90-θ

Solution.

$$(m+n)\cot Q = m\cdot\cot A - n\cdot\cot B$$

If $m = a$ and $n = a$, we can put them into our equation:

$$2a\cdot\cot Q = a\cdot\cot A - a\cdot\cot B$$

$$2a\cdot\cot Q = a(\cot A - \cot B)$$

$$2\cot Q = \cot A - \cot B$$

Substituted in equation B = α +2θ, A = β, Q = 90-θ:

$$2\cot(90-\theta) = \cot\beta - \cot(\alpha+2\theta)$$

$$\cot(90-\theta) = \tan\theta$$

$$2\tan\theta = \cot\beta - \cot(\alpha+2\theta)$$

$$\cot(\alpha+2\theta) = \frac{\cot\alpha\cot 2\theta - 1}{\cot 2\theta + \cot\alpha}$$

$$2\tan\theta = \cot\beta - \frac{\cot\alpha\cot 2\theta - 1}{\cot 2\theta + \cot\alpha}$$

$$2\tan\theta\,(\cot 2\theta + \cot\alpha) = \cot\beta\,(\cot 2\theta + \cot\alpha) - \cot\alpha\cot 2\theta + 1$$

$$\cot 2\theta = \frac{\cot\theta^2 - 1}{2\cot\theta}$$

$$2\tan\theta\left(\frac{\cot\theta^2 - 1}{2\cot\theta} + \cot\alpha\right) = \cot\beta\,(\cot 2\theta + \cot\alpha) - \cot\alpha\cot 2\theta + 1$$

$$\tan\theta^2\,(\cot\theta^2 - 1) + 2\tan\theta\cot\alpha = \cot 2\theta\,(\cot\beta - \cot\alpha) + \cot\beta\cot\alpha + 1$$

$$1 - \tan\theta^2 + 2\tan\theta\cot\alpha = \cot 2\theta\,(\cot\beta - \cot\alpha) + \cot\beta\cot\alpha + 1$$

$$\tan\theta^2 - 2\tan\theta\cot\alpha + \frac{\cot\theta^2 - 1}{2\cot\theta}(\cot\beta - \cot\alpha) + \cot\beta\cot\alpha = 0$$

Multiply by $\cot\theta$:

$$\tan\theta - 2\cot\alpha + \frac{(\cot\theta^2 - 1)}{2}(\cot\beta - \cot\alpha) + \cot\beta\cot\alpha = 0$$

$$\tan\theta - 2\cot\alpha + \frac{\cot\theta^2}{2}(\cot\beta - \cot\alpha) - 2(\cot\beta - \cot\alpha) + \frac{\cot\beta\cot\alpha}{\tan\theta} = 0$$

$$\tan\theta + \frac{\cot\beta\cot\alpha}{\tan\theta} + \frac{(\cot\beta - \cot\alpha)}{2\tan\theta^2} - 2\cot\beta = 0$$

$$\cot\theta^2 = x^2$$

$$\frac{(\cot\beta - \cot\alpha)}{2}x^2 + \cot\beta\cot\alpha\, x + \frac{1}{x} - 2\cot\beta = 0$$

Multiply by x:

$$\frac{(\cot\beta - \cot\alpha)}{2}x^3 + \cot\beta\cot\alpha\, x^2 - 2\cot\beta\, x + 1 = 0$$

If

$$\frac{(\cot\beta - \cot\alpha)}{2} = -a$$

$$\cot\beta\cot\alpha\, x^2 = b$$

$$2\cot\beta = c$$

Then

$$-ax^3 + bx^2 + cx + 1 = 0$$

$$x_1 = -\frac{\sqrt[3]{2}(-3ac-1)}{3a\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+ +27a^2+9ac+2}} + \frac{\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+27a^2+9ac+2}}{3\sqrt[3]{2}a} + \frac{1}{3a}$$

$$x_{2,3} = \frac{(1\pm i\sqrt{3})(-3ac-1)}{3\cdot 2^{\frac{2}{3}}a\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+ +27a^2+9ac+2}} - \frac{(1\mp i\sqrt{3})\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+ +27a^2+9ac+2}}{6\sqrt[3]{2}a} + \frac{1}{3a}$$

Answer:

$$x_1 = -\frac{\sqrt[3]{2}(-3ac-1)}{3a\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+ +27a^2+9ac+2}} + \frac{\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+27a^2+9ac+2}}{3\sqrt[3]{2}a} + \frac{1}{3a}$$

$$x_{2,3} = \frac{(1\pm i\sqrt{3})(-3ac-1)}{3\cdot 2^{\frac{2}{3}}a\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+ +27a^2+9ac+2}} - \frac{(1\mp i\sqrt{3})\sqrt[3]{\sqrt{(27a^2+9ac+2)^2+4(-3ac-1)^3}+ +27a^2+9ac+2}}{6\sqrt[3]{2}a} + \frac{1}{3a}$$

where

$\cot\theta \quad = x$

$$\frac{(\cot\beta - \cot\alpha)}{2} = -a$$

$$\cot\beta \cot\alpha \, x^2 = b$$

$$2\cot\beta \;= c$$