Given $V_{in}(t) = \sin \omega t$ and $V_{in} = V_c + V_r(out)$. Use laplace transform to show that

$$V_{out} = \frac{\omega RC}{1 + (\omega RC)^2} \cos \omega t + \frac{(\omega RC)^2}{1 + (\omega RC)^2} \sin \omega t - \frac{\omega RC}{1 + (\omega RC)^2} e^{-t/_{RC}}$$

Solution.

Start with Kirchhoff's loop law:

$$V_{in} = V_c + V_r$$

$$\sin \omega t = IR + {}^{Q}/_{C} = R\frac{dQ(t)}{dt} + \frac{Q(t)}{C}$$

We have differential equation:

$$R\dot{Q} + {}^{Q}/_{C} = \sin \omega t$$

Find the general solution of our differential equation:

$$R\dot{Q} + {}^{Q}/_{C} = 0$$

$$Q_{gs}(t) = e^{At}$$

$$RAe^{At} + {}^{e^{At}}/_{C} = 0$$

$$RA + {}^{1}/_{C} = 0$$

$$A = -{}^{1}/_{RC}$$

So

$$Q_{gs}(t) = e^{-1/_{RC}}$$

Find the particular solution of our differential equation:

$$Q_{ps}(t) = A \sin \omega t + B \cos \omega t$$

$$\sin \omega t = AR\omega \cos \omega t - BR\omega + {}^{A}/_{C} \sin \omega t + {}^{B}/_{C} \cos \omega t$$

Use laplace transform: $f(t) \risingdotseq F(s)$

$$\frac{\omega}{\omega^2 - s^2}\left(1 + \omega RB - {}^{A}/_{C}\right) + \frac{s}{\omega^2 - s^2}\left(-\omega RA - {}^{B}/_{C}\right) = 0$$

Find A and B:

$$A = \frac{\omega(RC)^2}{1 + (\omega RC)^2}$$

$$B = -\frac{RC}{1+(\omega RC)^2}$$

Then

$$q(s) = \frac{\omega(RC)^2}{1+(\omega RC)^2}\frac{\omega}{\omega^2 - s^2} - \frac{RC}{1+(\omega RC)^2}\frac{s}{\omega^2 - s^2} + \frac{\omega RC^2}{1+(\omega RC)^2}\frac{1}{(s+\frac{1}{RC})}$$

$Q(t) \risingdotseq q(s)$:

$$Q(t) = \frac{C}{1+(\omega RC)^2}\sin\omega t - \frac{\omega RC^2}{1+(\omega RC)^2}\cos\omega t + \frac{\omega RC^2}{1+(\omega RC)^2}e^{-t/_{RC}}$$

Whereas $V_{out} = V_r = IR = R\frac{dQ(t)}{dt}$, we have:

$$V_{out} = \frac{\omega RC}{1+(\omega RC)^2}\cos\omega t + \frac{(\omega RC)^2}{1+(\omega RC)^2}\sin\omega t - \frac{\omega RC}{1+(\omega RC)^2}e^{-t/_{RC}}$$