Task. If A be a subset of real number and b is real number then show that $\sup(b + A) = b + \sup(A)$.

Proof. By definition

$$x = \sup(A)$$

if $x \geq a$ for every $a \in A$, and for every $\varepsilon > 0$ there exists $a \in A$ such that

$$a > x - \varepsilon.$$

Also notice that

$$b + A = \{b + a \mid a \in A\}.$$

It suffices to prove that

$$\sup(b + A) \leq b + \sup(A).$$

Then applying this identity to $A' = b + A$ and $b' = -b$ we will obtain that

$$\sup(b' + A') \leq b' + \sup(A'),$$

that is

$$\sup(-b + b + A) \leq -b + \sup(b + A)$$
$$\sup(A) \leq -b + \sup(b + A),$$
$$\sup(A) + b \leq \sup(b + A).$$

Which will imply that

$$\sup(A) + b = \sup(b + A).$$

Let $x = \sup(A)$ and $y = \sup(b + A)$. We should prove that

$$y \leq b + x.$$

Suppose $y > b + x$. This means that there exist $\varepsilon > 0$ such that

$$y > y - \varepsilon > b + x.$$

But then there exist $z \in b + A$ such that

$$z > y - \varepsilon > b + x.$$

Notice that z has the fowm $z = b + a$ for some $a \in A$, whence

$$z = b + a > b + x$$

and so

$$a > x$$

which contradicts to the assumption that $x = \sup(A) \geq a$.

Hence $y \leq b + x$. And so

$$\sup(A) + b = \sup(b + A).$$