Show that the length of the curve $y = \log(secx)$ between the points $x = 0$ and $x = \frac{\pi}{3}$ is $log(2 + \sqrt{3})$

Explanation:

We have the given curve $y = \log(secx)$. Differentiating, we get:

$$\frac{dy}{dx} = \frac{1}{\sec(x)} = \sec(x)tan(x) = \tan(x)$$

Then, $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{ds}{dx}\right)^2 = 1 + tan^2(x) = sec^2(x)$

If we consider that the arc length S of the given curve is measured from $x = 0$ in the direction of x increasing, so we have:

$\frac{ds}{dx} = \sec(x)$ or we can write $ds = \sec(x)\,dx$

Accordingly if S_1denotes the arc length from $x = 0$ to $\frac{1}{3}\pi$, then

$$\int_0^{S_1} ds = \int_0^{\frac{1}{3}\pi} \sec(x)\,dx$$

We have function$f(x) = sec(x)$. We calculate the primitive (integral) for our function (constant arising in the integration, are not included here):

$$F(x) = \int f(x)dx = \int \sec(x)\,dx = \log(\sec(x) + \tan(x))$$

Finally we got:

$$F(x) = \log(\sec(x) + \tan(x))$$

By the theorem of Newton-Leibniz definite integral can be expressed as:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Substituting in the above formula our data, namely the primitive and the limits of integration:

$$\int_0^{S_1} ds = \int_0^{\frac{1}{3}\pi} \sec(x)\,dx = [\log(\sec(x) + \tan(x))]_0^{\frac{\pi}{3}}$$

$$= \left(\log\left(\sec\left(\frac{\pi}{3}\right) + \tan\left(\frac{\pi}{3}\right)\right)\right) - (\log(\sec(0) + \tan(0))) = \left(\log\left(2 + \sqrt{3}\right)\right) - (0)$$

$$= \log\left(2 + \sqrt{3}\right)$$

$S_1 = \log\left(2 + \sqrt{3}\right) \approx 1.31696$