For the function $f(x) = \csc x + \cot x$ on the interval $[0, 2pi]$

a) determine where it is discontinuous. Identify any vertical asymptotes

b) determine the intervals of concavity and inflection points

Solution.

a) First of all, let's transform our expression:

$$\csc x + \cot x = \frac{1}{\sin x} + \frac{\cos x}{\sin x} = \frac{1 + \cos x}{\sin x}$$

We can see that our function defined on the interval $[0, 2\pi]$ for all x except $\sin x \neq 0 \iff x \neq 0, 2\pi$, where it has a discontinuity.

So let's find one-sided limits at

$$\lim_{x \to 0-0} \frac{1 + \cos x}{\sin x} = \left[\frac{2}{-0}\right] = -\infty$$

$$\lim_{x \to 0+0} \frac{1 + \cos x}{\sin x} = \left[\frac{2}{+0}\right] = +\infty$$

$$\lim_{x \to 2\pi-0} \frac{1 + \cos x}{\sin x} = \left[\frac{2}{-0}\right] = -\infty$$

$$\lim_{x \to 2\pi+0} \frac{1 + \cos x}{\sin x} = \left[\frac{2}{+0}\right] = +\infty$$

The function $f(x)$ have a discontinuity of the second kind at $x = 0, 2\pi$, because the one-sided limits are infinite.

Vertical asymptotes are vertical lines which correspond to the zeroes of the denominator of a rational function.

So on the interval $[0, 2\pi]$ the denominator has zeros at $x = 0, 2\pi \Longrightarrow$

$\Longrightarrow$ we have two vertical asymptotes: $x = 0, x = 2\pi$.

b) we find the intervals of concavity and inflection points using the second derivative of the function $f(x) = \csc x + \cot x$.

$$f'(x) = (\csc x)' + (\cot x)' = -\cot x \csc x - \csc^2 x = -\csc x\,(\cot x + \csc x)$$

$$\begin{aligned} f'' = \left(-\csc x\,(\cot x + \csc x)\right)' &= -(\csc x)'(\cot x + \csc x) - \csc x\,(\cot x + \csc x)' \\ &= \cot x \csc x\,(\cot x + \csc x) + \csc^2 x\,(\cot x + \csc x) = \csc x\,(\cot x + \csc x)(\cot x + \csc x) \\ &= \csc x\,(\cot x + \csc x)^2 = \csc x \left(\frac{1 + \cos x}{\sin x}\right)^2 = \csc^3 x\,(1 + \cos x)^2 \end{aligned}$$

$$f'' = 0$$

$$\csc^3 x\,(1 + \cos x)^2 = 0$$

$$\csc x = 0 \Longrightarrow \textit{no solutions}$$

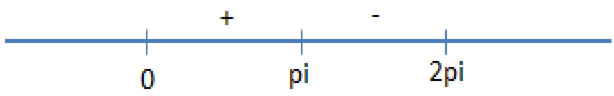
$$(1+\cos x)^2 = 0$$

$$1+\cos x = 0$$

$$\cos x = -1$$

$$x = \pi \quad on\ the\ interval\ x \in [0,2\pi]$$

$$x = \pi \quad - \quad inflection\ point$$



On the interval $(0,\pi)$ $\cos x > 0 \implies concave\ up$

On the interval $(\pi,2\pi)$ $\cos x < 0 \implies concave\ down$

Answer: $f(x)$ have a discontinuity at $x = 0, 2\pi$;

$f(x)$ have two vertical asymptotes: $x = 0, x = 2\pi$;

$f(x)$ $concave\ up$ on the interval $(0,\pi)$;

$f(x)$ $concave\ down$ on the interval $(\pi,2\pi)$;

$f(x)$ have inflection point: $x = \pi$