Question 1. *Let* $E \subseteq \mathbb{R}$ *be nonempty. Prove that* E *has an infimum if and only if* E *has a supremum, in which case* $\sup(-E) = -\inf E$.

Solution. Recall that $-E = \{-x \mid x \in E\}$. The set E has an infimum if and only if there is $a \in R$ such that $x \geq a$ for all $a \in E$, or, equivalently, $-x \leq -a$ for all $a \in E$. The latter means that $-E$ is bounded from above by $-a$, hence $-E$ has a supremum. We prove that $\sup(-E) = -a$. By definition of an infimum, for any $\varepsilon > 0$ there is $x_\varepsilon \in E$ such that $a \leq x_\varepsilon < a+\varepsilon$. Multiplying by -1, we get the inequality $-a \geq -x_\varepsilon > -a - \varepsilon$. Thus, for any $\varepsilon > 0$ there is $y_\varepsilon = -x_\varepsilon \in -E$ such that $-a - \varepsilon < y_\varepsilon \leq -a$. This is exactly the assertion that $-a = \sup(-E)$. □