Evaluate $\lim_{\theta \to \frac{\pi}{2}} \frac{\cot \theta - \cos \theta}{cos^3 \theta}$

Solution.

Take the limit:

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\frac{1}{\tan \theta} - \cos \theta}{cos^3 \theta}$$

Factor the numerator and denominator:

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{1 - \cos \theta \tan \theta}{\frac{\tan \theta \ (\cos^3 \theta \tan \theta)}{\tan \theta}}$$

Cancel terms, assuming $\frac{1}{\tan \theta} \neq 0$:

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{1 - \cos \theta \tan \theta}{\cos^3 \theta \tan \theta}$$

Indeterminate form of type 0/0. Applying L'Hospital's rule we have

$$\lim_{\theta \to \frac{\pi}{2}} \frac{1 - \cos \theta \tan \theta}{\cos^3 \theta \tan \theta} = \lim_{\theta \to \frac{\pi}{2}} \frac{\frac{d(1 - \cos \theta \tan \theta)}{d\theta}}{\frac{d(\cos^3 \theta \tan \theta)}{d\theta}}:$$

$$= \lim_{\theta \to \frac{\pi}{2}} \frac{2}{1 - 3 \cos 2\theta}$$

Factor out constants:

$$= 2(\lim_{\theta \to \frac{\pi}{2}} \frac{2}{1 - 3 \cos 2\theta})$$

The limit of a quotient is the quotient of the limits, the limit of constant is the constant:

$$= \frac{2}{\lim_{\theta \to \frac{\pi}{2}} (1 - 3 \cos 2\theta)}$$

The limit of sum is the sum of limits:

$$= \frac{2}{1 - 3 \lim_{\theta \to \frac{\pi}{2}} \cos 2\theta}$$

Using the continuity of $\cos \theta$ at $\theta = \pi$ write $\lim_{\theta \to \frac{\pi}{2}} \cos 2\theta$ as $\cos \left(\lim_{\theta \to \frac{\pi}{2}} 2\,\theta \right)$ and factor out constant:

$$= \frac{2}{1 - 3 \cos(2 \lim_{\theta \to \frac{\pi}{2}} \theta)} = \frac{2}{1 - 3 \cos(2 \cdot \frac{\pi}{2})} = \frac{1}{2}$$

Answer.

$$\lim_{\theta \to \frac{\pi}{2}} \frac{\cot \theta - \cos \theta}{cos^3\theta} = \mathbf{\frac{1}{2}}$$