Task. Use the Law of Cosines to find all the interior angles of a triangle having sides that measure 12.6 feet, 7.4 feet, and 8.2 feet. Use Heron's formula to find the area of the triangle.

Solution. Let $a = 12.6$, $b = 7.4$ and $c = 8.2$ be the sides of the triangle, and α, β, γ are the opposite angles.

Then the Law of Cosines claims that

$$c^2 = a^2 + b^2 - 2ab\cos\gamma,$$

whence

$$\gamma = \arccos\frac{a^2 + b^2 - c^2}{2ab}.$$

Similarly,

$$\beta = \arccos\frac{a^2 + c^2 - b^2}{2ac}.$$

$$\alpha = \arccos\frac{b^2 + c^2 - a^2}{2bc}.$$

Thus

$$\begin{aligned}\alpha &= \arccos\frac{b^2 + c^2 - a^2}{2bc} = \arccos\frac{7.4^2 + 8.2^2 - 12.6^2}{2 * 7.4 * 8.2} = \arccos\frac{-36.760}{121.36} \\ &= \arccos(-0.30290) = 107.632^\circ\end{aligned}$$

$$\begin{aligned}\beta &= \arccos\frac{a^2 + c^2 - b^2}{2ac} = \arccos\frac{12.6^2 + 8.2^2 - 7.4^2}{2 * 12.6 * 8.2} = \arccos\frac{171.24}{206.64} \\ &= \arccos(0.82869) = 34.036^\circ\end{aligned}$$

$$\begin{aligned}\gamma &= \arccos\frac{a^2 + b^2 - c^2}{2ab} = \arccos\frac{12.6^2 + 7.4^2 - 8.2^2}{2 * 12.6 * 7.4} = \arccos\frac{146.28}{186.48} \\ &= \arccos(0.78443) = 38.332^\circ\end{aligned}$$

Now let us find the area of the triangle using Heron's formula:

$$S = \sqrt{p(p-a)(p-b)(p-c)},$$

where

$$p = \frac{a + b + c}{2}$$

is the half-perimeter. Thus

$$p = \frac{12.6 + 7.4 + 8.2}{2}\frac{28.2}{2} = 14.1 \text{ feet}.$$

Then

$$\begin{aligned}S &= \sqrt{14.1 * (14.1 - 12.6) * (14.1 - 7.4) * (14.1 - 8.2)} \\ &= \sqrt{14.1 * 1.5 * 6.7 * 5.9} = \sqrt{863.06} \approx 28.915 \text{ feet}^2.\end{aligned}$$