Find the Taylor series

$$x + \frac{3}{(x-1)(x-4)} \ at \ x = 2$$

Solution.

Divide $\frac{3}{(x-1)(x-4)}$ into two summands:

$$\frac{3}{(x-1)(x-4)} = \frac{A}{x-1} + \frac{B}{x-4}$$

$$3 = Ax - 4A + Bx - B$$

$$x(A+B) + (-4A - B - 3) = 0$$

$$\begin{cases} A + B = 0 \\ 4A + B + 3 = 0 \end{cases}$$

$$\begin{cases} A = -1 \\ B = 1 \end{cases}$$

$$x + \frac{3}{(x-1)(x-4)} = x + \frac{1}{x-4} - \frac{1}{x-1}$$

$$\frac{1}{x-4} = \frac{1}{(x-2)-2} = \frac{-1/2}{1 - \frac{x-2}{2}} = -\frac{1}{2} \cdot \sum_{n=0}^{\infty} \left(\frac{x-2}{2}\right)^n = -\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (x-2)^n \quad for \ \left|\frac{x-2}{2}\right| < 1 \ \Leftrightarrow \ 0 < x < 4$$

$$\frac{1}{x-1} = \frac{1}{1+(x-2)} = \sum_{n=0}^{\infty} (-1)^n (x-2)^n \quad for \ |2-x| < 1 \ \Leftrightarrow \ 1 < x < 3$$

Then

$$x + \frac{3}{(x-1)(x-4)} = 2 + (x-2) + \sum_{n=0}^{\infty} (-1)^n (x-2)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+1} (x-2)^n =$$

$$= 2 + (x-2) + \sum_{n=0}^{\infty} \left[(-1)^n - \left(\frac{1}{2}\right)^{n+1}\right] \cdot (x-2)^n \quad for \ 1 < x < 3$$

Answer:

$$2 + (x-2) + \sum_{n=0}^{\infty} \left[(-1)^n - \left(\frac{1}{2}\right)^{n+1}\right] (x-2)^n \quad for \ 1 < x < 3$$

Find the Taylor series

$$\frac{x+3}{(x-1)(x-4)}\ at\ x=2$$

Solution.

$$(x+3)\cdot\frac{1}{(x-1)(x-4)}=(x+3)\cdot\frac{1}{3}\left[\frac{1}{x-4}-\frac{1}{x-1}\right]=\frac{1}{3}\cdot\left(\frac{x+3}{x-4}-\frac{x+3}{x-1}\right)$$

$$\frac{x+3}{x-4}=\frac{x-4+7}{x-4}=1+\frac{7}{x-4}=1-\sum_{n=0}^{\infty}7\cdot\left(\frac{1}{2}\right)^{n+1}(x-2)^n\quad for\ \left|\frac{x-2}{2}\right|<1\ \Leftrightarrow\ 0<x<4$$

$$\frac{x+3}{x-1}=\frac{x-1+4}{x-1}=1+\frac{4}{x-1}=1+\sum_{n=0}^{\infty}4\cdot(-1)^n(x-2)^n\quad for\ |2-x|<1\ \Leftrightarrow\ 1<x<3$$

Then

$$\frac{1}{3}\cdot\left(\frac{x+3}{x-4}-\frac{x+3}{x-1}\right)=\frac{1}{3}\cdot\left(2+\sum_{n=0}^{\infty}\left[4\cdot(-1)^n-7\cdot\left(\frac{1}{2}\right)^{n+1}\right](x-2)^n\right)\quad for\ 1<x<3$$

Answer:

$$\frac{2}{3}+\sum_{n=0}^{\infty}\left[\frac{4}{3}\cdot(-1)^n-\frac{7}{3}\cdot\left(\frac{1}{2}\right)^{n+1}\right](x-2)^n\quad for\ 1<x<3$$