Question 1. *Let D be a non-empty subset of the real numbers, and $E = \{ax \mid x \in D\}$ where $a > 0$. Prove that E is open if and only if D is open.*

Solution. It is sufficient to prove the implication: if D is open, then E is open (the converse implication is proved by applying the result for $\frac{1}{a} > 0$, because $D = \{\frac{x}{a} \mid x \in E\}$).

Let $x_0 \in E$, i. e. $x_0 = ay_0$ for some $y_0 \in D$. Since D is open, there is $\varepsilon > 0$ such that all y satisfying $y_0 - \varepsilon < y < y_0 + \varepsilon$ belong to D. Then $ay \in E$ for all such y. Note that $x_0 - a\varepsilon < ay < x_0 + a\varepsilon$ and, moreover, any z from the interval $(x_0 - a\varepsilon, x_0 + a\varepsilon)$ can be expressed as $z = ay$, where $y \in (y_0 - \varepsilon, y_0 + \varepsilon) \subseteq D$. Thus, $(x_0 - a\varepsilon, x_0 + a\varepsilon) \subseteq E$, which proves that E is open. $\square$