How to find the two tangent to the Hyperbola $y = 1/x$ passing through the point $P(2, -4)$,

Show that the straight line $y = mx + b$ is a tangent to $y = -x^2 - 3x + 4$ if $(m + 3)^2 = 4(b - 4)$

Solution:

1. The equation of the line passing through the point $P(2, -4)$ is

$$(y + 4) = k(x - 2)$$

Let (x_0, y_0) is the point on the Hyperbola.

The slope of the tangent line to the Hyperbola at $x = x_0$ is

$y'_0 = -\frac{1}{{x_0}^2}$, so

$$(y + 4) = -\frac{1}{{x_0}^2}(x - 2)$$

When $x = x_0$ then $y = \frac{1}{x_0}$

$$\left(\frac{1}{x_0} + 4\right) = -\frac{1}{{x_0}^2}(x_o - 2)$$

$$x_0 + {x_0}^2 = -x_0 + 2$$

$${x_0}^2 + 2x_0 - 2 = 0$$

$$x_0 = -1 - \sqrt{3}, x_0 = -1 + \sqrt{3}$$

Hence there are two tangents to the Hyperbola $y = 1/x$ passing through the point $P(2, -4)$:

$$(y + 4) = -\frac{1}{\left(-1 - \sqrt{3}\right)^2}(x - 2)$$

$$(y + 4) = -\frac{1}{4 + 2\sqrt{3}}(x - 2)$$

And

$$(y + 4) = -\frac{1}{\left(-1 + \sqrt{3}\right)^2}(x - 2)$$

$$(y+4) = -\frac{1}{4-2\sqrt{3}}(x-2)$$

2. If line $y = mx + b$ is a tangent to $y = -x^2 - 3x + 4$ at the point (x_0, y_0), then

$m = y'_0 = -2x_0 - 3$ and $y_0 = -x_0^2 - 3x_0 + 4$, so

$$-x_0^2 - 3x_0 + 4 = (-2x_0 - 3)x_0 + b$$

$$-x_0^2 - 3x_0 + 4 = -2x_0^2 - 3x_0 + b$$

So

$$b - 4 = x_0^2$$

$$m + 3 = -2x_0$$

$$(m+3)^2 = (-2x_0)^2 = 4x_0^2$$

Hence

$$(m+3)^2 = 4(b-4)$$