Task. The function f is such that $f(x) = (2\sin x)^2 - (3\cos x)^2$ for $0 \le x \le \pi$.
i) express $f(x)$ in the form $a + b(\cos x)^2$ stating the values of a and b;
ii) state the greatest and least values of $f(x)$
iii) solve the equation $f(x) + 1 = 0$
Solution.
i) Recall that for any x we have that

$$(\sin x)^2 + (\cos x)^2 = 1.$$

Therefore

$$(\sin x)^2 = 1 - (\cos x)^2,$$

whence

$$\begin{aligned} f(x) &= (2\sin x)^2 - (3\cos x)^2 = 4(\sin x)^2 - 9(\cos x)^2 \\ &= 4(1 - (\cos x)^2) - 9(\cos x)^2 \\ &= 4 - 4(\cos x)^2 - 9(\cos x)^2 = 4 - 13(\cos x)^2, \end{aligned}$$

so

$$f(x) = a + b(\cos x)^2,$$

where

$$a = 4, \qquad b = -13.$$

ii) Notice that the maximal value of $(\cos x)^2$ is 1, whence the maximal value of f is

$$\max f = 4 - 13 \cdot 1 = -9.$$

iii) Let us solve the equation

$$f(x) + 1 = 0,$$

that is

$$\begin{gathered} 4 - 13(\cos x)^2 + 1 = 0, \\ 13(\cos x)^2 = 5 \\ (\cos x)^2 = 5/13, \\ \cos x = \pm\sqrt{5/13}. \end{gathered}$$

Since $x \in [0, \pi]$, it follows that

$$\begin{aligned} \cos x = \sqrt{5/13} \qquad &\Rightarrow \qquad x = \arccos\sqrt{5/13}, \\ \cos x = -\sqrt{5/13} \qquad &\Rightarrow \qquad x = \pi - \arccos\sqrt{5/13}. \end{aligned}$$

So we have two solutions:

$$\arccos\sqrt{5/13}, \qquad \pi - \arccos\sqrt{5/13}.$$