Question #29040

$$\lim_{x \to x_0} f(x) = A \quad \Leftrightarrow \quad \forall \varepsilon > 0 \ \ \exists \delta > 0 \ \ \forall x: 0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - A| < \varepsilon$$

(for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - A| < \varepsilon$)

Prove: $\lim_{x \to -4} x^2 = 16$

$$|f(x) - A| < \varepsilon \Rightarrow |x^2 - 16| < \varepsilon$$

$$|(x - 4)(x + 4)| < \varepsilon$$

$$|x + 4| < \frac{\varepsilon}{|x - 4|}$$

Now we can let δ equal $\frac{\varepsilon}{|x-4|}$. But what do we do about the $|x - 4|$. In general δ must be

in terms of ε only, without any extra variables. So how we can remove this x -4 term? First we need to simplify the problem a little bit. Since the concept of limit only applies when x is close to x_0, we will first restrict x so that it is at most 1 away from x_0, or, mathematically, in our case,

$|x + 4| < 1$. Then, this means, $-5 < x < -3$, or , $-9 < x - 4 < -7$. $7 < |x - 4| < 9$

Now consider the original inequality $|x + 4| < \frac{\varepsilon}{|x-4|} < \frac{\varepsilon}{7}$

Since we now have two restriction, $|x + 4| < 1$ and $|x + 4| < \frac{\varepsilon}{7}$ we let $\delta = \min\{1, \frac{\varepsilon}{7}\}$, the smaller of these two values, which guarantees that it will satisfy both inequalities. Finally, after all this, we can write up the proof. Given ε, let $\delta = \min\{1, \frac{\varepsilon}{7}\}$.

Suppose $\delta = 1$. Since $1 < \frac{\varepsilon}{7}$, we know $\varepsilon > 7$. Then:

$$|x - x_0| < \delta \Rightarrow |x + 4| < 1 \Rightarrow |(x - 4)(x + 4)| < |x - 4| \Rightarrow |x^2 - 16| < |x - 4| < 9 < \varepsilon$$

$$\Rightarrow |x^2 - 16| < \varepsilon \Rightarrow |f(x) - A| < \varepsilon$$

This completes the case $\delta = 1$.

Now suppose $\delta = \frac{\varepsilon}{7}$. Then $|x - x_0| < \delta \Rightarrow |x + 4| < \frac{\varepsilon}{7} \Rightarrow |(x - 4)(x + 4)| < \frac{9\varepsilon}{7}$

$$\Rightarrow |x^2 - 16| < \frac{9\varepsilon}{7} < \varepsilon \Rightarrow |f(x) - A| < \varepsilon$$

Finally, this completes the proof.