Task. The events X and Y are independent. Calculate the $P(X \cap Y)$ if $P(X) = 0.64$ and $P(Y) = 0.23$. Hence find $P(X \cup Y)$.

Solution. We can regard events X and Y as subsets of some probability space Ω. Then the event "X and Y" corresponds to the intersection $X \cap Y$, while the event "X or Y" is the union $X \cup Y$ of these events.

Recall that events X and Y are called *independent* if

$$P(X \cap Y) = P(X) \cdot P(Y).$$

Thus in our case

$$P(X \text{ and } Y) = P(X \cap Y) = P(X) \cdot P(Y) = 0.64 \cdot 0.23 = 0.1472.$$

We should compute $P(X \cup Y)$. We claim that

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y).$$

Indeed, recall that is the intersection $A \cap B$ of events A and B is empty, then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B).$$

Notice that $X \cup Y$ can be represented as a union of three disjoint subsets:

$$X \cup Y = (X \setminus Y) \ \cup \ (X \cap Y) \ \cup \ (Y \setminus X),$$

whence

$$P(X \cup Y) = P(X \setminus Y) + P(X \cap Y) + P(Y \setminus X).$$

Moreover,

$$X = (X \setminus Y) \ \cup \ (X \cap Y), \qquad Y = (Y \setminus X) \ \cup \ (X \cap Y),$$

whence

$$P(X) = P(X \setminus Y) + P(X \cap Y), \qquad P(Y) = P(Y \setminus X) + P(X \cap Y).$$

Therefore

$$P(X \setminus Y) = P(X) - P(X \cap Y), \qquad P(Y \setminus X) = P(Y) - P(X \cap Y),$$

and so

$$P(X \cup Y) = P(X) - P(X \cap Y) + P(X \cap Y) + P(Y) - P(X \cap Y) = P(X) + P(Y) - P(X \cap Y).$$

Thus

$$P(X \cup Y) = P(X) + P(Y) - P(X \cap Y) = 0.64 + 0.23 - 0.1472 = 0.7228.$$