Question 1. *Please show that any convergent sequence in a metric space is bounded, then find a bounded sequence of real numbers that is not convergent.*

Solution. Let $x_n \to a$ as $n \to \infty$ in a metric space (X, ρ). By definition of convergence for any $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that $\rho(x_n, a) < \varepsilon$ for all $n > N$. Take $\varepsilon = 1$ and find the corresponding $N \in \mathbb{N}$. Set M to be $\max\{\rho(x_1, a), \ldots, \rho(x_N, a), \varepsilon\}$. Then for all $n \geq 1$ we have $\rho(x_n, a) \leq M$. Thus, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is contained in the closed ball of radius M with the center in a. So, $\{x_n\}_{n\in\mathbb{N}}$ is bounded.

For an example of a bounded real sequence which is not convergent consider $x_n = (-1)^n$. We have $|x_n| = 1$ for all n, but there are two subsequences $x_{2n} = 1$ and $x_{2n-1} = -1$, $n \geq 1$, which are obviously convergent and have different limits. This explains why x_n cannot be convergent. $\square$