Determine whether $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\dots$ converges or diverges.

Solution:

$$1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\dots=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-2}-\frac{1}{4n-1}-\frac{1}{4n}\right)=$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n-1}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{4n-2}-\frac{1}{4n}\right)=\sum_{n=1}^{\infty}\frac{2}{(4n-3)(4n-1)}+\sum_{n=1}^{\infty}\frac{2}{(4n-2)4n}$$

Limit comparison test

Suppose that we have two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ with $a_n, b_n \geq 0$ for all n.

Then if $\lim_{n\to\infty}\frac{a_n}{b_n}=c$ with $0 < c < \infty$ then either both series converge or both series diverge.

Comparing with the convergent series $\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\lim_{n\to\infty}\frac{\frac{2}{(4n-3)(4n-1)}}{\frac{1}{n^2}}=\frac{2n^2}{16n^2-16n+3}=\frac{1}{8}$$

$$\lim_{n\to\infty}\frac{\frac{2}{(4n-2)4n}}{\frac{1}{n^2}}=\frac{2n^2}{16n^2-8n}=\frac{1}{8}$$

So $\sum_{n=1}^{\infty}\frac{2}{(4n-3)(4n-1)}$ and $\sum_{n=1}^{\infty}\frac{2}{(4n-2)4n}$ converge

If $\sum_{n=1}^{\infty}\frac{2}{(4n-3)(4n-1)}$ and $\sum_{n=1}^{\infty}\frac{2}{(4n-2)4n}$ converge, so their sum converges.

Hence $1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots$ converges.