Task. Find the area of the triangle having medians of length 9 cm, 12 cm and 15 cm.

Solution. Let a, b, c be the sides of the triangle, and

$$m_a = 9, \qquad m_b = 12, \qquad m_c = 15$$

be the medians to the corresponding sides.

We will use the formula for the area of the triangle:

$$S = \frac{1}{2} ab \sin\gamma,$$

where γ is the angle between a and b.

First we obtain a formula expressing the side of the triangle via its medians, and then using cosine theorem will find $\cos\gamma$ and $\sin\gamma$.

Let us recall the formula for the median:

$$m_c^2 = \frac{1}{2}\sqrt{2(a^2+b^2)-c^2},$$

whence

$$4m_c^2 = 2(a^2+b^2)-c^2.$$

Similarly,

$$4m_a^2 = 2(b^2+c^2)-a^2,$$
$$4m_b^2 = 2(a^2+c^2)-b^2.$$

Adding these equations we will get

$$4(m_a^2+m_b^2+m_c^2) = 2(a^2+b^2)-c^2+2(b^2+c^2)-a^2+2(a^2+c^2)-b^2,$$
$$4(m_a^2+m_b^2+m_c^2) = 3(a^2+b^2+c^2).$$

From

$$4m_a^2 = 2(b^2+c^2)-a^2$$

we also obtain that

$$b^2+c^2 = 2m_a^2 + a^2/2,$$

whence

$$4(m_a^2+m_b^2+m_c^2) = 3(a^2+2m_a^2+a^2/2),$$
$$4(m_a^2+m_b^2+m_c^2) = 6m_a^2+9a^2/2,$$
$$9a^2/2 = 4m_a^2+4m_b^2+4m_c^2-6m_a^2,$$
$$9a^2/2 = 4m_b^2+4m_c^2-2m_a^2,$$

and so

$$a^2 = \frac{4}{9}(2m_b^2+2m_c^2-m_a^2) = \frac{4}{9}(2*12^2+2*15^2-9^2) = 292.$$

Similarly,

$$b^2 = \frac{4}{9}(2m_a^2+2m_c^2-m_b^2) = \frac{4}{9}(2*9^2+2*15^2-12^2) = 208,$$
$$c^2 = \frac{4}{9}(2m_a^2+2m_b^2-m_c^2) = \frac{4}{9}(2*9^2+2*12^2-15^2) = 100.$$

Let γ be the angle between sides a and c. Then by cosine theorem

$$c^2 = a^2+b^2-2ab\cos\gamma,$$

whence

$$\cos\gamma = \frac{a^2+b^2-c^2}{2ab} = \frac{292+208-100}{2\sqrt{292*208}} = \frac{400}{2\sqrt{292*208}} =$$
$$= \frac{200}{\sqrt{64*13*73}} = \frac{200}{8\sqrt{13*73}} = \frac{25}{\sqrt{949}}.$$

Therefore

$$\sin\gamma = \sqrt{1-\cos^2\alpha} = \sqrt{1-\frac{25^2}{949}} = \sqrt{\frac{949-625}{949}} = \frac{324}{\sqrt{949}} = \frac{18}{\sqrt{949}},$$

and so the area of the triangle is equal to

$$S = \frac{1}{2}ab\sin\gamma = \frac{\sqrt{292*208}}{2}\cdot\frac{18}{\sqrt{949}} = \frac{8\sqrt{949}}{2}\cdot\frac{18}{\sqrt{949}} = 4*18 = 72cm^2.$$