$$\begin{cases} \frac{dy}{dx} = xz + 1 \\ \frac{dz}{dx} = -xy \end{cases},$$

y(0)=0, z(0)=1, x=0.3

Solution

Solve this system with a step h=0.3, $\varphi(x, y, z) = xz + 1, \psi(x, y, z) = -xy$, $x_0 = 0, y_0=0, z_0 = 1$

$y_{0.3} = y_{0+0.3} = y_0 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$,

$$z_{0.3} = z_{0+0.3} = z_0 + \frac{h}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$

$k_1 = \varphi(x_0, y_0, z_0) = 0 \cdot 1 + 1 = 1$

$l_1 = \psi(x_0, y_0, z_0) = 0$

$k_2 = \varphi\left(x_0 + {}^{h}/_{2}, y_0 + {}^{hk_1}/_{2}, z_0 + {}^{hl_1}/_{2}\right) = 1.15$

$l_2 = \psi\left(x_0 + {}^{h}/_{2}, y_0 + {}^{hk_1}/_{2}, z_0 + {}^{hl_1}/_{2}\right) = -0.0225$

$k_3 = \varphi\left(x_0 + {}^{h}/_{2}, y_0 + {}^{hk_2}/_{2}, z_0 + {}^{hl_2}/_{2}\right)$ =1.14949375

$l_3 = \psi\left(x_0 + {}^{h}/_{2}, y_0 + {}^{hk_2}/_{2}, z_0 + {}^{hl_2}/_{2}\right) = -0.025875$

$k_4 = \varphi(x_0 + h, y_0 + hk_3, z_0 + hl_3) = 1.29767125$

$l_4 = \psi(x_0 + h, y_0 + hk_3, z_0 + hl_3) = -0.1034544375$

$$y_{0.3} = 0 + \frac{0.3}{6}(1 + 2 \cdot 1.15 + 2 \cdot 1.14949375 + 1.29767125) = 0.3448329375$$

$$z_{0.3} = 1 + \frac{0.3}{6}(0 + 2 \cdot (-0.0225) + 2 \cdot (-0.025875) - 0.1034544375) = 0.989989778125$$

General solution:

From the second equation: $y = -\frac{1}{x}\frac{dz}{dx}$, let differentiate it changing $\frac{dy}{dx} = xz + 1$:

$xz + 1 = \frac{1}{x^2}\frac{dz}{dx} - \frac{1}{x}\frac{d^2z}{dx^2}$,

$$\frac{1}{x}\frac{d^2z}{dx^2} - \frac{1}{x^2}\frac{dz}{dx} + xz + 1 = 0$$

Let $t = \frac{x^2}{2}$, so $x = \sqrt{2t}$:

$$\frac{\frac{d^2z}{dx^2}}{\sqrt{2t}} + \sqrt{2t}z - \frac{\frac{dz}{dx}}{2t} + 1 = 0,$$

By the chain rule $\frac{dz}{dx} = \frac{dt}{dx}\frac{dz}{dt}$ and $\frac{d^2z}{dx^2} = (\frac{dt}{dx})^2\frac{d^2z}{dt^2} + \frac{d^2t}{dx^2}\frac{dz}{dt}$.

Therefore $\frac{dz}{dx} = \sqrt{2t}\frac{dz}{dt}$ and $\frac{d^2z}{dx^2} = 2t\frac{d^2z}{dt^2} + \frac{dz}{dt}$.

Substitute these values into the differential equation:

$$\frac{2t\frac{d^2z}{dt^2}+\frac{dz}{dt}}{\sqrt{2t}}+\sqrt{2t}z-\frac{\frac{dz}{dt}}{\sqrt{2t}}+1=0.$$

Simplify and divide both sides by $\sqrt{2t}$:

$$\frac{d^2z}{dt^2}+z+\frac{1}{\sqrt{2t}}=0.$$

The general solution is a sum of the complementary and particular solutions.

Find complementary solution from: $\frac{d^2z}{dt^2}+z=0$. Assume a solution will be proportional to $e^{\lambda t}$ for some constant λ:

$$\frac{d^2}{dt^2}\left(e^{\lambda t}\right)+e^{\lambda t}=0.$$

Substitute $\frac{d^2}{dt^2}\left(e^{\lambda t}\right)=\lambda^2 e^{\lambda t}, so\ \lambda^2 e^{\lambda t}+e^{\lambda t}=0,$

$e^{\lambda t}(\lambda^2+1)=0$, such as $e^{\lambda t}\neq 0$ then $\lambda^2+1=0$. Solve it: $\lambda=\pm i$.

So $z_c=c_1\cos(t)+c_2\sin(t)$.

Determine the particular solution to $\frac{d^2z}{dt^2}+z=-\frac{1}{\sqrt{2t}}$ by variation of parameters.

List the basis solutions in z_c: $z_{b1}=\cos(t)$ and $z_{b2}=\sin(t)$.

Compute the Wronskian of z_{b1} and z_{b2}:

$$W(t)=\begin{vmatrix}\cos(t) & \sin(t)\\ \frac{d\cos(t)}{dt} & \frac{d\sin(t)}{dt}\end{vmatrix}=\begin{vmatrix}\cos(t) & \sin(t)\\ -\sin(t) & \cos(t)\end{vmatrix}=1.$$

Let $f(t)=-\frac{1}{\sqrt{2t}}$;

let $v_1(t)=-\int\frac{f(t)z_{b2}}{W(t)}dt$ and $v_2(t)=-\int\frac{f(t)z_{b1}}{W(t)}dt$.

The particular solution will be given by:

$$z_p=v_1(t)z_{b1}+v_2(t)z_{b2}.$$

$$v_1(t)=-\int\frac{-\sin(t)}{\sqrt{2t}}dt=\sqrt{\pi}S(\sqrt{\frac{2t}{\pi}}),\ v_2(t)=\int\frac{-\cos(t)}{\sqrt{2t}}dt=-\sqrt{\pi}C(\sqrt{\frac{2t}{\pi}}).$$

So $z_p=\sqrt{\pi}S\left(\sqrt{\frac{2t}{\pi}}\right)\cos(t)-\sqrt{\pi}C\left(\sqrt{\frac{2t}{\pi}}\right)\sin(t)=\sqrt{\pi}(S\left(\sqrt{\frac{2t}{\pi}}\right)\cos(t)-C\left(\sqrt{\frac{2t}{\pi}}\right)\sin(t)).$

The general solution:

$$z=z_c+z_p=c_1\cos(t)+c_2\sin(t)+\sqrt{\pi}(S\left(\sqrt{\frac{2t}{\pi}}\right)\cos(t)-C\left(\sqrt{\frac{2t}{\pi}}\right)\sin(t)).$$

Substitute back for $t=\frac{x^2}{2}$:

$z = c_1 \cos\left(\frac{x^2}{2}\right) + c_2 \sin\left(\frac{x^2}{2}\right) + \sqrt{\pi}(S\left(\frac{x}{\sqrt{\pi}}\right)\cos\left(\frac{x^2}{2}\right) - C\left(\frac{x}{\sqrt{\pi}}\right)\sin\left(\frac{x^2}{2}\right)).$

$\frac{dz}{dx} =$
$-c_1 \,\mathrm{xsin}\left(\frac{x^2}{2}\right) + c_2 \,\mathrm{xcos}\left(\frac{x^2}{2}\right) - \sin\left(\frac{x^2}{2}\right)C'\left(\frac{x}{\sqrt{\pi}}\right) - \sqrt{\pi}\,\mathrm{xcos}\left(\frac{x^2}{2}\right)C\left(\frac{x}{\sqrt{\pi}}\right) + +\cos\left(\frac{x^2}{2}\right)S'\left(\frac{x}{\sqrt{\pi}}\right) -$
$\sqrt{\pi}\,\mathrm{xsin}\left(\frac{x^2}{2}\right)S\left(\frac{x}{\sqrt{\pi}}\right).$

So $y = -\frac{1}{x}\frac{dz}{dx} = c_1 \sin\left(\frac{x^2}{2}\right) - c_2 \cos\left(\frac{x^2}{2}\right) + \frac{1}{x}\sin\left(\frac{x^2}{2}\right)C'\left(\frac{x}{\sqrt{\pi}}\right) + \sqrt{\pi}\cos\left(\frac{x^2}{2}\right)C\left(\frac{x}{\sqrt{\pi}}\right) -$
$-\frac{1}{x}\cos\left(\frac{x^2}{2}\right)S'\left(\frac{x}{\sqrt{\pi}}\right) + \sqrt{\pi}\sin\left(\frac{x^2}{2}\right)S\left(\frac{x}{\sqrt{\pi}}\right).$

Answer

$y_{0.3} = 0.3448329375$, $z_{0.3} = 0.989989778125.$

$z = c_1 \cos\left(\frac{x^2}{2}\right) + c_2 \sin\left(\frac{x^2}{2}\right) + \sqrt{\pi}(S\left(\frac{x}{\sqrt{\pi}}\right)\cos\left(\frac{x^2}{2}\right) - C\left(\frac{x}{\sqrt{\pi}}\right)\sin\left(\frac{x^2}{2}\right)).$

$y =$
$c_1 \sin\left(\frac{x^2}{2}\right) - c_2 \cos\left(\frac{x^2}{2}\right) + \frac{1}{x}\sin\left(\frac{x^2}{2}\right)C'\left(\frac{x}{\sqrt{\pi}}\right) + \sqrt{\pi}\cos\left(\frac{x^2}{2}\right)C\left(\frac{x}{\sqrt{\pi}}\right) - -\frac{1}{x}\cos\left(\frac{x^2}{2}\right)S'\left(\frac{x}{\sqrt{\pi}}\right) +$
$\sqrt{\pi}\sin\left(\frac{x^2}{2}\right)S\left(\frac{x}{\sqrt{\pi}}\right).$