Task. A fair coin is tossed 99 times. If x is the number of times heads occurs then when $P(x = r)$ is maximum?

Answer. Notice that x has binominal distribution with $n = 99$, $p = q = 0.5$. Therefore the probability

$$P(x = r) = C_n^r p^r q^{n-r},$$

where

$$C_n^r = \frac{n!}{r!\,(n-r)!}.$$

Since $p = q = 0.5$ we obtain that

$$P(x = r) = C_n^r 0.5^r 0.5^{n-r} = 0.5^n\, C_n^r.$$

Thus maximum of $P(x = r)$ corresponds to the value of r with maximal

$$C_n^r = \frac{n!}{r!\,(n-r)!}.$$

Notice that binominal coefficients are symmetric:

$$C_n^r = C_n^{n-r} = \frac{n!}{r!\,(n-r)!},$$

whence the maximum corresponds to the points most closest to $n/2$.

Since n is odd, there are two such values:

$$r_1 = \frac{n-1}{2} = \frac{98}{2} = 49, \qquad r_2 = \frac{n+1}{2} = \frac{100}{2} = 50.$$

For these numbers the function $P(x = r)$ has maximum.