How to determine area and volume of the function $f(x) = 2 + \sin(x)$ rotated around the x axis from $(0; 2\pi)$? The slice is a circle.

Solution:

Volume

$$V = \pi \int_0^{2\pi} (f(x))^2 dx = \pi \int_0^{2\pi} (2 + \sin(x))^2 dx = \pi \int_0^{2\pi} (4 + 4\sin(x) + (\sin(x))^2) dx =$$

$$= \pi \left(4x|_0^{2\pi} - 4\cos(x)|_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} (1 - \cos(2x)) dx \right) =$$

$$= \pi \left(8\pi - 0 + \frac{1}{2}x|_0^{2\pi} - \frac{1}{4}\sin(2x)|_0^{2\pi} \right) = \pi(8\pi - 0 + \pi - 0) = 9\pi^2 (cubic\ units)$$

Area

$$S = 2\pi \int_0^{2\pi} f(x)\sqrt{1 + \left(f'(x)\right)^2} dx = 2\pi \int_0^{2\pi} (2 + \sin(x))\sqrt{1 + (\cos(x))^2} dx =$$

$$= 4\pi \int_0^{2\pi} \sqrt{1 + (\cos(x))^2} dx = 16\pi \int_0^{\frac{\pi}{2}} \sqrt{1 + (\cos(x))^2} dx$$

We have to replacement

$$t = \tan\left(\frac{x}{2}\right), dx = \frac{2dt}{1 + t^2}, \cos(x) = \frac{1 - t^2}{1 + t^2}$$

Then we have

$$S = 16\pi \int_0^1 \sqrt{1 + \left(\frac{1 - t^2}{1 + t^2}\right)^2} \frac{2dt}{1 + t^2} = 32\pi \int_0^1 \frac{\sqrt{(1 + t^2)^2 + (1 - t)^2} dt}{(1 + t^2)^2} =$$

$$= 32\sqrt{2}\pi \int_0^1 \frac{\sqrt{1 + t^4} dt}{(1 + t^2)^2}$$

This integral is very difficult. So we can use numerical integration. Then we have

$$S \approx 32\sqrt{2}\pi \cdot 0.678551 \approx 96.47113\ (square\ units)$$