Please solve the following IBVP:

u_tt = u_xx  on 0<x<pi, t>0.
u(0,t) = 0, t >= 0.
u(pi,t) = 0, t>= 0.
u(x,0) = sin^3(x) 0<=x<=pi.
u_t(x,0) = sin(2x) 0<=x<=pi.

Solution:

The general solution of this IBVP is

$$u(x,t) = \sum_{k=1}^{\infty} (A_k \cos(kt) + B_k \sin(kt))\sin(kx)$$

where

$A_k = \frac{2}{\pi}\int_0^{\pi} \varphi(x)\sin(kx)\,dx$, $B_k = \frac{2}{k\pi}\int_0^{\pi} \psi(x)\sin(kx)\,dx$, $\varphi(x) = sin^3(x)$, $\psi(x) = \sin(2x)$.

Then we have

$$B_k = \frac{2}{k\pi}\int_0^{\pi} \sin(2x)\cdot\sin(kx)\,dx = \begin{cases} \frac{1}{2}, k = 2; \\ 0, k \neq 2. \end{cases}$$

$$A_k = \frac{2}{\pi}\int_0^{\pi} sin^3(x)\sin(kx)\,dx$$

$$A_1 = \frac{2}{\pi}\int_0^{\pi} sin^4(x)dx = \frac{2}{\pi}\left(-\frac{3}{8}\sin(x)\cos(x) - \frac{1}{4}sin^3(x)\cos(x) + \frac{3}{8}x\right)\Big|_0^{\pi} = \frac{3}{4};$$

$$A_2 = \frac{2}{\pi}\int_0^{\pi} sin^3(x)\sin(2x)\,dx = \frac{4}{\pi}\int_0^{\pi} sin^4(x)\cos(x)\,dx =$$

$$= \frac{4}{\pi}\int_0^{\pi} sin^4(x)\,\mathrm{d}(\sin(x)) = \frac{4}{5\pi}\left(sin^5(x)\right)\Big|_0^{\pi} = 0;$$

$$A_3 = \frac{2}{\pi}\int_0^{\pi} sin^3(x)\sin(3x)\,dx = \frac{2}{\pi}\int_0^{\pi} sin^3(x)\left(3\sin(x) - 4sin^3(x)\right)dx =$$

$$= \frac{6}{\pi}\int_0^{\pi} sin^4(x)dx - \frac{8}{\pi}\int_0^{\pi} sin^6(x)dx = \frac{6}{\pi}\cdot\frac{3}{8}\pi -$$

$$-\frac{8}{\pi}\left(-\frac{1}{6}sin^5(x)\cos(x)-\frac{5}{16}\sin(x)\cos(x)-\frac{5}{24}sin^3(x)\cos(x)+\frac{5}{16}x\right)\Bigg|_0^\pi =$$

$$=\frac{9}{4}-\frac{8}{\pi}\cdot\frac{5}{16}\pi=\frac{9}{4}-\frac{5}{2}=-\frac{1}{4};$$

$$A_k=\frac{2}{\pi}\int_0^\pi sin^3(x)\sin(kx)\,dx=0\ if\ k>3.$$

So we have the next solution

$$u(x,t)=\sum_{k=1}^{\infty}(A_k\cos(kt)+B_k\sin(kt))\sin(kx)=$$

$$=\frac{1}{2}\sin(2t)\sin(2x)+\frac{3}{4}\cos(t)\sin(x)-\frac{1}{4}\cos(3t)\sin(3x).$$