**Find the canonical form of the following PDE:**

$$\boldsymbol{u_{xx} - 6u_{xy} + 9u_{yy} = xy^2}$$

****Be sure to show the change of coordinates that reduces the PDE to canonical form.**

**Solution:**

$a = 1, b = -3, c = 9$

$\Delta = b^2 - ac = (-3)^2 - 1 \cdot 9 = 0$, so the equation is parabolic.

Write down the characteristic equation:

$$dy^2 + 6dxdy + 9dx^2 = 0$$

$$\frac{dy}{dx} = -3$$

$$y = -3x + C$$

Determine new variables $\xi$ and $\eta$ :

$$\xi = \varphi(x, y) = 3x + y$$

$$\varphi_x = 3, \varphi_y = 1, \varphi_{xx} = 0, \varphi_{xy} = 0, \varphi_{yy} = 0$$

$$\eta = \psi(x, y) = x$$

$$\psi_x = 1, \psi_y = 0, \psi_{xx} = 0, \psi_{xy} = 0, \psi_{yy} = 0$$

$$\begin{vmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} \neq 0$$

$$u_{xx} = u_{\xi\xi}\varphi_x{}^2 + 2u_{\xi\eta}\varphi_x\psi_x + u_{\eta\eta}\psi_x{}^2 + u_\xi\varphi_{xx} + u_\eta\psi_{xx} = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = u_{\xi\xi}\varphi_x\varphi_y + u_{\xi\eta}(\varphi_x\psi_y + \varphi_y\psi_x) + u_{\eta\eta}\psi_x\psi_y + u_\xi\varphi_{xy} + u_\eta\psi_{xy} = 3u_{\xi\xi} + u_{\xi\eta}$$

$$u_{yy} = u_{\xi\xi}\varphi_y{}^2 + 2u_{\xi\eta}\varphi_y\psi_y + u_{\eta\eta}\psi_y{}^2 + u_\xi\varphi_{yy} + u_\eta\psi_{yy} = u_{\xi\xi}$$

$$9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta} - 6(3u_{\xi\xi} + u_{\xi\eta}) + 9u_{\xi\xi} = \eta(\xi - 3\eta)^2$$

The canonical form is:

$$u_{\eta\eta} = \eta(\xi - 3\eta)^2$$