Bolzano–Weierstrass theorem

In real analysis, the Bolzano–Weierstrass theorem is a fundamental result about convergence in a finite-dimensional Euclidean space R^n. The theorem states that each bounded sequence in R^n has a convergent subsequence. An equivalent formulation is that a subset of R^n is sequentially compact if and only if it is closed and bounded.

Proof

First we prove the theorem in the case of n = 1, in which case the ordering on R can be put to good use. Indeed we have the following result.

Rising Sun Lemma: Every sequence $\{x_n\}$ in R has a monotone subsequence.

Proof: Let us call a positive integer "n" a peak of the sequence if m > n implies $x_n > x_m$, i.e., if x_n is greater than every subsequent term in the sequence. Suppose first that the sequence has infinitely many peaks. Then the subsequence of peaks is strictly decreasing, and we are done. So suppose now that there are only finitely many peaks, and let "N" be the last peak. By definition, this means that there is n_1 > N with $x_{n_1} < x_n$. Since n_1 > N, n_1 is not a peak, which, as above, implies the existence of an $n_2 > n_1$ with $x_{n_2} < x_{n_1}$. Repeating this process leads to an infinite subsequence $x_{n_1} < x_{n_2}$.

Now suppose we have a bounded sequence in R; by the Rising Sun Lemma there exists a monotone subsequence, necessarily bounded. But it is immediate from the completeness of the real numbers that any bounded nondecreasing (respectively, nonincreasing) sequence converges to its least upper bound (respectively, greatest lower bound).

Finally, the general case can be easily reduced to the case of n=1 as follows: given a bounded subsequence in R^n, the sequence of first coordinates is a bounded real sequence, so, has a convergent subsequence. We can then extract a subsubsequence on which the second coordinates converge, and so on, until in the end we have passed from the original sequence to a subsequence "n" times -- which is still a subsequence of the original sequence -- on which each coordinate sequence converges, so, the subsequence itself is convergent.