Show that $f(x)=x^2$ is not uniformly continuous on R.

Solution.

Function $f(x)$ is uniformly continuous on R if for arbitrary $\varepsilon > 0$ there exists $\delta > 0$ such that as soon as $|x - x_0| < \delta$ it follows that $|f(x) - f(x_0)| < \varepsilon$ for any $x_0 \in R$.

Note that the value of δ doesn't depend upon the choice of x_0.

Now suppose that $\varepsilon > 0$ is given and let us try to find corresponding value of δ.

Using Lagrange theorem we can write that

$$|f(x) - f(x)_0| = |f'(c)| * |x - x_0| = |2c||x - x_0|,$$

where $c \in [x_0, x]$.

Let $|x - x_0| < \delta$ and in order to find the value of δ we demand that

$$|2c||x - x_0| < |2c|\delta = \varepsilon.$$

From this $\delta = \varepsilon / |2c|$.But as the value of c depends on x_0 then the value of δ depends on the choice of x_0 too. So the value of δ which is common for all points $x_0 \in R$ doesn't exist. Therefore the function $f(x) = x^2$ is not uniformly continuous on R. Q.E.D.