Use Green's theorem to evaluate

$$\oint_C \left((x^2 + xy)dx + (x^2 + y^2)dy\right)$$

where C is the square formed by the lines y=±1 and x±1.

Solution:

Green's theorem:

Let *C* be a positively oriented, piecewise smooth, simple closed curve in a plane, and let *D* be the region bounded by *C*. If *L* and *M* are functions of (*x*, *y*) defined on an open region containing *D* and have continuous partial derivatives there, then

$$\oint_C (L(x,y)dx + M(x,y)dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right) dxdy$$

where the path of integration along C is counterclockwise.

In our case

$$L(x,y) = x^2 + xy, M(x,y) = x^2 + y^2,$$

$$\frac{\partial M}{\partial x} = 2x, \frac{\partial L}{\partial y} = x.$$

Thus

$$\oint_C \left((x^2 + xy)dx + (x^2 + y^2)dy\right) = \iint_D (2x - x)dxdy =$$

$$= \iint_D xdxdy = \int_{-1}^{1} dy \int_{-1}^{1} xdx = \int_{-1}^{1} \left(\frac{x^2}{2}\Big|_{-1}^{1}\right) dy = \int_{-1}^{1} \left(\frac{1^2}{2} - \frac{(-1)^2}{2}\right) dy =$$

$$= \int_{-1}^{1} \left(\frac{1}{2} - \frac{1}{2}\right) dy = 0$$

Answer:

$$\oint_C \left((x^2 + xy)dx + (x^2 + y^2)dy\right) = 0$$