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Determine and classify the extreme values of $u = x^2y^2 - 5x^2 - 8xy - 5y^2$

Solution:

To find critical points we solve the following equation

$$du = 0$$

$$du = 2xy^2dx + 2x^2ydy - 10xdx - 8ydx - 8xdy - 10ydy = 0$$

$$(2xy^2 - 10x - 8y)dx + (2x^2y - 10y - 8x)dy = 0$$

$$\begin{cases} 2xy^2 - 10x - 8y = 0 \\ 2x^2y - 10y - 8x = 0 \end{cases} \to \begin{cases} x(y^2 - 5) - 4y = 0 \\ x^2y - 5y - 4x = 0 \end{cases}$$

$$x = \frac{4y}{(y^2 - 5)}$$

$$\frac{16y^3}{(y^2 - 5)^2} - 5y - \frac{16y}{(y^2 - 5)} = 0$$

$$y = 0$$

$$\frac{16y^2}{(y^2 - 5)^2} - 5 - \frac{16}{(y^2 - 5)} = 0$$

$$16y^2 - 5(y^2 - 5)^2 - 16(y^2 - 5) = 0$$

$$-5(y^2 - 5)^2 + 80 = 0$$

$$(y^2 - 5)^2 = 16$$

$$y^2 - 5 = \pm 4$$

$y^2 = 1$ and $y^2 = 9$

$$y = -1, y = 1, y = -3, y = 3$$

When $y = 0$, then $x = 0$

When $y = -1$, then $x = \frac{-4}{(1-5)} = 1$

When $y = 1$, then $x = \frac{4}{(1-5)} = -1$

When $y = -3$, then $x = \frac{-12}{(9-5)} = -3$

When $y = 3$, then $x = \frac{12}{(9-5)} = 3$

We found five critical points: $(0, 0), (1, -1), (-1,1), (-3, -3), (3,3)$.

$$\frac{\partial^2 u}{\partial^2 x} = 2y^2 - 10$$

$$\frac{\partial^2 u}{\partial^2 y} = 2x^2 - 10$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = 4xy - 8$$

At the first point we have:

$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial^2 x} \times \frac{\partial^2 u}{\partial^2 y} = (-8)^2 - (-10)(-10) = -36 < 0$ and $\frac{\partial^2 u}{\partial^2 x} = -10 < 0$, so

(0,0) is the maximum.

At the second and third point it is:

$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial^2 x} \times \frac{\partial^2 u}{\partial^2 y} = (-12)^2 - (-8)(-8) = 84 > 0$, so the points $(1, -1), (-1, 1)$ are not extreme.

At the fourth and fifth points it is:

$\left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 - \frac{\partial^2 u}{\partial^2 x} \times \frac{\partial^2 u}{\partial^2 y} = (28)^2 - 8 \times 8 = 720 > 0$, so the points $(-3, -3), (3,3)$ are not extreme.

Answer: (0,0) is the maximum.