Explain tensor. prove that the sum of two tensor is a tensor. Show that by contraction , the rank of tensor is reduced by two.

Solution

$A^{i_1 ... i_n}_{i_{n+1} ... i_{n+m}}, B^{i_1 ... i_n}_{i_{n+1} ... i_{n+m}}$ is tensors of (n,m)-type.

According to the transformation law of tensors we have

$$\widehat{A}^{i_1 ... i_n}_{i_{n+1} ... i_{n+m}} = (R)^{-1}{}^{i_1}_{j_1} ... (R)^{-1}{}^{i_n}_{j_n} R^{j_{n+1}}_{i_{n+1}} ... R^{j_{n+m}}_{i_{n+m}} A^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}}$$

$$\widehat{B}^{i_1 ... i_n}_{i_{n+1} ... i_{n+m}} = (R)^{-1}{}^{i_1}_{j_1} ... (R)^{-1}{}^{i_n}_{j_n} R^{j_{n+1}}_{i_{n+1}} ... R^{j_{n+m}}_{i_{n+m}} B^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}}$$

Where $R_i{}^j$ is a matrix**.** The components, v^i, of a regular (or column) vector, **v**, transform with the inverse of the matrix *R*,

$$\widehat{v}^i = (R)^{-1}{}^i_j v^j$$

where the hat denotes the components in the new basis. While the components, w_i, of a covector (or row vector), **w** transform with the matrix R itself,

$$\widehat{w}_i = R^j_i w_j$$

$$\Rightarrow$$

$$\alpha A^{i_1 ... i_n}_{j_1 ... j_k} + \beta B^{i_1 ... i_n}_{j_1 ... j_k} = (R)^{-1}{}^{i_1}_{j_1} ... (R)^{-1}{}^{i_n}_{j_n} R^{j_{n+1}}_{i_{n+1}} ... R^{j_{n+m}}_{i_{n+m}} (\alpha A^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}} + \beta B^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}})$$

α, β is constant

From hence $\alpha A^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}} + \beta B^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}}$ transformed as tensor.

From hence $\alpha A^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}} + \beta B^{j_1 ... j_n}_{j_{n+1} ... j_{n+m}}$ is tensor

Contraction is $A^{i_1 ... i_n}_{i_1 ... i_{n+m}}$.

$$\widehat{A}^{i_1 ... i_n}_{i_1 ... i_{n+m}} = (R)^{-1}{}^{i_1}_{j_1} ... (R)^{-1}{}^{i_n}_{j_n} R^{j_1}_{i_1} ... R^{j_{n+m}}_{i_{n+m}} A^{j_1 ... j_n}_{j_1 ... j_{n+m}} \Rightarrow$$

$$\widehat{A}^{i_1 ... i_n}_{i_1 ... i_{n+m}} = (R)^{-1}{}^{i_2}_{j_2} ... (R)^{-1}{}^{i_n}_{j_n} R^{j_{n+2}}_{i_{N=2}} ... R^{j_{n+m}}_{i_{n+m}} A^{j_1 ... j_n}_{j_1 ... j_{n+m}}$$

Because $(R)^{-1}{}_{j_1}^{i_1} * R_{i_1}^{j_1} = E$ is identity matrix.

From hence $A_{j_1 \ldots j_{n+m}}^{j_1 \ldots j_n}$ transformed as tensor (n-1,m-1) rank.

From hence $A_{j_1 \ldots j_{n+m}}^{j_1 \ldots j_n}$ is (n-1,m-1) rank.