Question:

If the roots of the equation $x^3 + 3px^2 + 3qx + r = 0$ are in HP, show that $2q^3 - 3pqr + r^2 = 0$

Solution:

The given equation is:

$$x^3 + 3px^2 + 3qx + r = 0 \text{ (1)}$$

Given that roots of the equation in H.P. Changing x to $\frac{1}{x}$:

$$\frac{1}{x^3} + \frac{3p}{x^2} + \frac{3q}{x} + r = 0$$

Multiplying by x^3, we get:

$$1 + 3px + 3qx^2 + rx^3 = 0 \text{ (2)}$$

Roots of (2) being reciprocals of the roots of (1) must be in A.P. Let the roots of (2) be $\alpha - \beta, \alpha, \alpha + \beta$.

From equation (2) sum of roots is equal = $-\frac{3q}{r}$, so

$$(\alpha - \beta) + \alpha + (\alpha + \beta) = -\frac{3q}{r}$$

$$3\alpha = -\frac{3q}{r}$$

$$\alpha = -\frac{q}{r}$$

α is root of (2), so

$$1 + 3p\left(-\frac{q}{r}\right) + 3q\left(-\frac{q}{r}\right)^2 + r\left(-\frac{q}{r}\right)^3 = 0$$

$$1 - \frac{3pq}{r} + \frac{3q^3}{r^2} - \frac{q^3}{r^2} = 0$$

$$1 - \frac{3pq}{r} + \frac{2q^3}{r^2} = 0$$

$$r^2 - 3pqr + 2q^3 = 0, \text{ so}$$

$$2q^3 - 3pqr + r^2 = 0$$