Show that $n^2 + n + 1$ is not divisible by 5 for any n.

Solution:
Any natural number is divisible by 5 with remainders 0,1,2,3,4.
for remainders 0 1 and 2 we can provide numbers as $5k, 5k + 1$ and $5k + 2$
for remaider 3 we can provide numbers as $5k + 3 = 5k + 5 - 2 = 5(k + 1) - 2$ (remainder -2)
for remaider 4 we have the same situation $5k + 4 = 5k + 5 - 1 = 5(k + 1) - 1$ (remainder -1)

So any natural number we can provide in the form of

$$n = 5k$$
$$n = 5k \pm 1$$
$$n = 5k \pm 2$$

where k natural number.

Case $n = 5k$ then
$$n^2 + n + 1 = (5k)^2 + 5k + 1 = 25k^2 + 5k + 1 = 5k(5k + 1) + 1$$
not divisible by 5 (remainder 1).

Case $n = 5k \pm 1$ then
$$n^2 + n + 1 = 25k^2 \pm 10k + 1 + 5k \pm 1 + 1 = 5k(5k \pm 1) + 2 \pm 1$$
also not divisible by 5 (remainder 3 or 1).

Case
$$n = 5k \pm 2$$
$$n^2 + n + 1 = 25k^2 \pm 10k + 4 + 5k \pm 2 + 1 = 5k(5k \pm 1 + 1) \pm 2$$

also not divisible by 5 (remainder 2 or 3)
So $n^2 + n + 1$ isn’t divided by 5 for any n