Solve the Cauchy problem on a semi-infinite domain:

$U_{tt} - 4U_{xx} = 0$

$U(x,0) = xe^{-x} = \varphi(x)$

$U_t(x,0) = 0 = \psi(x)$

$U(0,t) = 0 = \mu(t)$

$0 \le x < +\infty; t \ge 0$

Solution:

a) $x - 2t \ge 0$

$$U(x,t) = \frac{\varphi(x-2t) + \varphi(x+2t)}{2} + \frac{1}{4}\int_{x-2t}^{x+2t} \psi(y)dy = \frac{(x-2t)e^{2t-x} + (x+2t)e^{x+2t}}{2}$$

b) $x - 2t < 0$

$$U(x,t) = \frac{\varphi(x+2t) - \varphi(2t-x)}{2} + \frac{1}{4}\int_{2t-x}^{x+2t} \psi(y)dy = \frac{(x+2t)e^{-(x+2t)} - (2t-x)e^{x-2t}}{2}$$

Answer:

$$U(x,t) = \begin{cases} \dfrac{(x-2t)e^{2t-x} + (x+2t)e^{x+2t}}{2}, & when\, x \ge 2t \\ \dfrac{(x+2t)e^{-(x+2t)} - (2t-x)e^{x-2t}}{2}, & when\, x < 2t \end{cases}$$