Question 1. *Show that any ideal (resp. subring) can be realized as the kernel (resp. the image) of a ring homomorphism?*

Solution. Let I be an ideal of a ring R. Consider the natural projection $j : R \to R/I$, which maps any $r \in R$ to $a + I \in R/I$. Since the zero of R/I is the class $0 + I = I$, we have

$$r \in \ker j \Leftrightarrow j(r) = I \Leftrightarrow r + I = I \Leftrightarrow r \in I.$$

Thus, $\ker j = I$.

Now let S be a subring of R. Then there is an embedding i of S into R, namely, $i(s) = s \in S \subseteq R$ for all $s \in S$. Obviously, $\operatorname{im} i = S$. □