Task:

Prove Heron's formula.

Solution:

Heron's formula relates the area, A, of a triangle with the half perimeter, s:

$A = \sqrt{s(s-a)(s-b)(s-c)}$[1.1]

where s=(a+b+c)/2, and a, b, c are the lengths of the sides.

The following proof is trigonometric, and basically uses the cosine rule. First we compute the cosine squared in terms of the sides, and then the sine squared which we use in the formula $A = \frac{1}{2}bc \cdot sin\alpha$ to derive the area of the triangle in terms of its sides, and thus prove Heron's formula.

We use the relationship $x^2 - y^2 = (x+y)(x-y)$ [difference between two squares] [1.2]

From the cosine rule:

$a^2 = b^2 + c^2 - 2bc\cos\alpha$

We have:

$2bc\cos\alpha = b^2 + c^2 - a^2$ [1.3]

Rearranging:

$\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$ [1.4]

Because we want the sine, we first square the cosine:

$\cos^2\alpha = \frac{\left(b^2+c^2-a^2\right)^2}{(2bc)^2}$ [1.5]

Finding the Sine

To use in:

$\sin^2\alpha = 1 - \cos^2\alpha$ [1.6]

Using Equation 1.5 in 1.6, we have:

$\sin^2\alpha = 1 - \frac{\left(b^2+c^2-a^2\right)^2}{(2bc)^2}$ [1.7]

Bringing all under the same denominator:

$\sin^2\alpha = \frac{(2bc)^2 - \left(b^2+c^2-a^2\right)^2}{(2bc)^2}$ [1.8]

Using the difference between two squares (Equation 1.2)

$$\sin^2 \alpha = \frac{(b^2+c^2-a^2+2bc)(2bc-b^2-c^2+a^2)}{(2bc)^2} \text{ [1.9]}$$

Putting the above into a form where we can use the difference between two squares again we have:

$$\sin^2 \alpha = \frac{((b+c)^2-a^2)(a^2-(b-c)^2)}{(2bc)^2} \text{ [1.10]}$$

Actually using the difference between two squares in both brackets, we find:

$$\sin^2 \alpha = \frac{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}{(2bc)^2} \text{ [1.11]}$$

Substituting $(a + b + c)$ for $2s$, $(b + c - a)$ for $2s - 2a$, etc:

$$\sin^2 \alpha = \frac{2s(2s-2a)(2s-2c)(2s+2b)}{(2bc)^2} \text{ [1.12]}$$

Taking the square root:

$$\sin \alpha = \sqrt{\frac{2s(2s-2a)(2s-2c)(2s+2b)}{(2bc)^2}} \text{ [1.13]}$$

<u>Finding the Area</u>

Recalling:

$$A = \frac{1}{2} bc \cdot \sin \alpha \text{ [1.14]}$$

We have:

$$A = \frac{bc}{2} \sqrt{\frac{2s(2s-2a)(2s-2c)(2s+2b)}{(2bc)^2}} \text{ [1.15]}$$

And simplified:

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

which is Heron's formula.