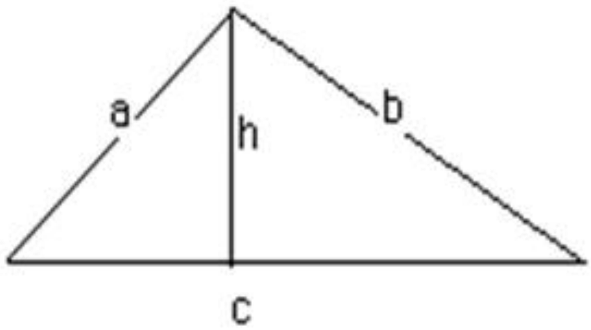
prove Heron's formula for area of a triangle whose sides are given a,b,c

The demonstration and proof of Heron's formula can be done from elementary consideration of geometry and algebra. I will assume the Pythagorean theorem and the area formula for a triangle

$$A = \frac{1}{2}bh,$$

where b is the length of a base and h is the height to that base.

Let a, b, and c be the lengths of the sides of our triangle and h the height to the side of length c.



We have

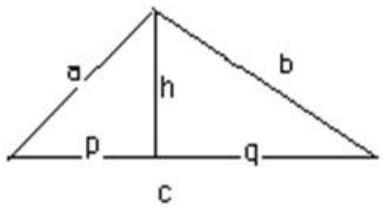
$$S = \frac{a+b+c}{2},$$

so, for future reference,

$$2s = a + b + c$$
$$2(s - a) = - a + b + c$$
$$2(s - b) = a - b + c$$
$$2(s - c) = a + b - c$$

There is at least one side of our triangle for which the altitude lies "inside" the triangle. For convenience make that the side of length c. It will not make any difference, just simpler.

Our task is to express h in terms of a, b, and c, then substitute for h in $A = \frac{1}{2}ch$.



Let p + q = c as indicated. Then

$$h^2 + p^2 = a^2$$

and

$$h^2 + q^2 = b.$$

Since

$$q = c - p,$$

then

$$q^2 = (c - p)^2$$

and

$$q^2 = c^2 - 2cp + p^2.$$

Adding h^2 to each side gives

$$h^2 + q^2 = h^2 + c^2 - 2cp + p^2.$$

Now substitute in this equation and get $b^2 = a^2 - 2cp + c^2$ and solve for p to get

$$p = \frac{a^2 + c^2 - b^2}{2c}.$$

Now, since $h^2 = a^2 - p^2$ we can substitute for p and get an expression in terms of a, b, and c.

$$h^2 = a^2 - p^2 =$$

$$(a + p)(a - p) =$$

$$\left(a + \frac{a^2 + c^2 - b^2}{2c}\right)\left(a - \frac{a^2 + c^2 - b^2}{2c}\right) =$$

$$\left(\frac{2ac + a^2 + c^2 - b^2}{2c}\right)\left(\frac{2ac - a^2 - c^2 + b^2}{2c}\right) =$$

$$\frac{((a + c)^2 - b^2)(b^2 - (a - c)^2)}{4c^2} = \frac{(a + b + c)(-a + b + c)(a + b + c)(a + b - c)}{4c^2} =$$

$$\frac{2s * 2(s-a) * 2(s-b) * 2(s-c)}{4c^2}$$

Therefore

$$h^2 = \frac{4s(s-a)(s-b)(s-c)}{c^2}$$

$$h^2 = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{c}$$

Since

$$A = \frac{1}{2}ch,$$

then

$$A = \frac{1}{2}c\frac{2\sqrt{s(s-a)(s-b)(s-c)}}{c},$$

and

$$A = \sqrt{s(s-a)(s-b)(s-c)}.$$