$$\sin\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}+\frac{1}{3}\sin\frac{3\pi}{3}+\cdots+\frac{1}{n}\sin\frac{n\pi}{3}+\cdots$$

Solution:

We have

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{3}=$$

$$=\sum_{m=0}^{+\infty}\left(\frac{1}{3m+1}\sin\frac{(3m+1)\pi}{3}+\frac{1}{3m+2}\sin\frac{(3m+2)\pi}{3}+\frac{1}{3m+3}\sin\frac{(3m+3)\pi}{3}\right)=$$

$$=$$

$$=\sum_{m=0}^{+\infty}\left(\frac{1}{3m+1}\sin(m\pi+\frac{\pi}{3})+\frac{1}{3m+2}\sin((m+1)\pi-\frac{\pi}{3})+\frac{1}{3m+3}\sin((m+1)\pi)\right)=$$

$$=\sum_{m=0}^{+\infty}\left(\frac{1}{3m+1}(-1)^m\frac{\sqrt{3}}{2}+\frac{1}{3m+2}(-1)^m\frac{\sqrt{3}}{2}+\frac{1}{3m+3}\cdot 0\right)$$

$$=\frac{\sqrt{3}}{2}\sum_{m=0}^{+\infty}(-1)^m\left(\frac{1}{3m+1}+\frac{1}{3m+2}\right)$$

So we have

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{3}=\frac{\sqrt{3}}{2}\sum_{m=0}^{+\infty}(-1)^m\left(\frac{1}{3m+1}+\frac{1}{3m+2}\right)$$

Denote

$$a_m=(-1)^m\left(\frac{1}{3m+1}+\frac{1}{3m+2}\right)$$

Because $\frac{1}{3(m+1)+1}<\frac{1}{3m+1}$ *and* $\frac{1}{3(m+1)+2}<\frac{1}{3m+2}$ *then* $|a_{m+1}|<|a_m|$.

Because $\lim_{m\to\infty}|a_m|=0$ *then series*

$$\frac{\sqrt{3}}{2}\sum_{m=0}^{+\infty}(-1)^m\left(\frac{1}{3m+1}+\frac{1}{3m+2}\right)$$

*is convergent. But because h*armonic series

$$\sum_{m=0}^{+\infty}\frac{1}{m}$$

is divergent t*hen*

$$\sum_{n=1}^{+\infty}\frac{1}{n}\left|\sin\frac{n\pi}{3}\right|\to+\infty$$

Then the series

$$\sin\frac{\pi}{3}+\frac{1}{2}\sin\frac{2\pi}{3}+\frac{1}{3}\sin\frac{3\pi}{3}+\cdots+\frac{1}{n}\sin\frac{n\pi}{3}+\cdots$$

is **conditionally convergent** because it is convergent, but not absolutely convergent.