Given $h \in G$, let $\varphi : kG \to kG$ be the linear map defined by $\varphi(\alpha) = h\alpha e$. Consider the decomposition $kG = kG \cdot e \oplus kG \cdot (1 - e)$.

On $kG \cdot e$, φ is just left multiplication by h, and on $kG \cdot (1 - e)$, φ is the zero map. Therefore, $\chi(h) = \mathrm{tr}(\varphi)$. Now $\varphi(\alpha) = h\alpha \sum_{g \in G} a_g \ g = \sum_{g \in G} a_g \ h\alpha g$.

Thus, if $\varphi_g : kG \to kG$ is defined by $\varphi_g(\alpha) = h\alpha g$, we have $\chi(h) = \sum_{g \in G} a_g \ \mathrm{tr}(\varphi_g)$. We finish by calculating $\mathrm{tr}(\varphi_g)$ $(g \in G)$. Since φ_g takes G to G, $\mathrm{tr}(\varphi_g)$ is just the number of group elements $\alpha \in G$ such that $h\alpha g = \alpha$, i.e. $g = \alpha^{-1}h^{-1}\alpha$. Thus, if g is not conjugate to h^{-1}, $\mathrm{tr}(\varphi_g) = 0$. If g is conjugate to h^{-1}, the number of $\alpha \in G$ such that $g = \alpha^{-1}h^{-1}\alpha$ is $|C_G(h)|$, so $\mathrm{tr}(\varphi_g) = |C_G(h)|$. It follows that $\chi(h) = |C_G(h)| \cdot \sum_{g \in C} a_g$, where C is the conjugacy class of h^{-1} in G.