QUESTION:
Find the equation of the line through (12/5,1), forming with the axes a triangle of area 5.

SOLUTION:

Let's write an equation of straight line in slope-intercept form:

$y = m\cdot x + b$

This line crosses y-axis at the point $(0;b)$ and x-axis at the point $(-\frac{b}{m};0)$. Hence, it forms with the axes a right triangle. The area of this triangle is $S = \frac{1}{2}\cdot|b|\cdot\left|\frac{b}{m}\right|$. We use here the absolute values of slope and intercept, because they could be negative.

So, as line go through the point $(\frac{12}{5},1)$, and area of triangle is 5, we can write a system of equations:

$$\begin{cases} m\cdot\frac{12}{5}+b=1 \\ \frac{1}{2}|b|\cdot\left|\frac{b}{m}\right|=5 \end{cases} \Rightarrow \begin{cases} 12\cdot m+5b=5 \\ |b|\cdot\left|\frac{b}{m}\right|=10 \end{cases} \Rightarrow \begin{cases} 12\cdot m+5b=5 \\ \frac{b^2}{|m|}=10 \end{cases}$$

Let's consider two cases:

a) $m>0$, hence $|m|=m$ and:

$$\begin{cases} 12\cdot m+5b=5 \\ \frac{b^2}{m}=10 \end{cases} \Rightarrow \begin{cases} 12\cdot m+5b=5 \\ m=\frac{b^2}{10} \end{cases} \Rightarrow \begin{cases} 12\cdot\frac{b^2}{10}+5b=5 \\ m=\frac{b^2}{10} \end{cases} \Rightarrow \begin{cases} \frac{6}{5}\cdot b^2+5b=5 \\ m=\frac{b^2}{10} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} 6\cdot b^2+25b-25=0 \\ m=\frac{b^2}{10} \end{cases} \Rightarrow \begin{cases} b_1=\frac{-25+\sqrt{25^2-4\cdot 6\cdot(-25)}}{2\cdot 6} \\ b_2=\frac{-25-\sqrt{25^2-4\cdot 6\cdot(-25)}}{2\cdot 6} \\ m_1=\frac{b_1^2}{10} \\ m_2=\frac{b_1^2}{10} \end{cases} \Rightarrow \begin{cases} b_1=\frac{-25+35}{12} \\ b_2=\frac{-25-35}{12} \\ m_1=\frac{b_1^2}{10} \\ m_2=\frac{b_1^2}{10} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} b_1=\frac{5}{6} \\ b_2=-5 \\ m_1=\frac{5}{72} \\ m_2=\frac{5}{2} \end{cases}$$

b) $m < 0$, hence $|m| = -m$ and:

$$\begin{cases} 12 \cdot m + 5b = 5 \\ \dfrac{b^2}{-m} = 10 \end{cases} \Rightarrow \begin{cases} 12 \cdot m + 5b = 5 \\ m = \dfrac{-b^2}{10} \end{cases} \Rightarrow \begin{cases} -12 \cdot \dfrac{b^2}{10} + 5b = 5 \\ m = \dfrac{-b^2}{10} \end{cases} \Rightarrow \begin{cases} -\dfrac{6}{5} \cdot b^2 + 5b = 5 \\ m = \dfrac{-b^2}{10} \end{cases}$$

$$\Rightarrow \begin{cases} 6 \cdot b^2 - 25b + 25 = 0 \\ m = \dfrac{-b^2}{10} \end{cases} \Rightarrow \begin{cases} b_3 = \dfrac{25 + \sqrt{25^2 - 4 \cdot 6 \cdot 25}}{2 \cdot 6} \\ b_4 = \dfrac{-25 - \sqrt{25^2 - 4 \cdot 6 \cdot 25}}{2 \cdot 6} \\ m_3 = \dfrac{-b_3^2}{10} \\ m_4 = \dfrac{-b_4^2}{10} \end{cases} \Rightarrow \begin{cases} b_3 = \dfrac{25+5}{12} \\ b_4 = \dfrac{25-5}{12} \\ m_1 = \dfrac{b_3^2}{10} \\ m_2 = \dfrac{b_4^2}{10} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} b_3 = \dfrac{5}{2} \\ b_4 = \dfrac{5}{3} \\ m_3 = -\dfrac{5}{8} \\ m_4 = -\dfrac{5}{18} \end{cases}$$

So, we’ve obtained four straight lines:

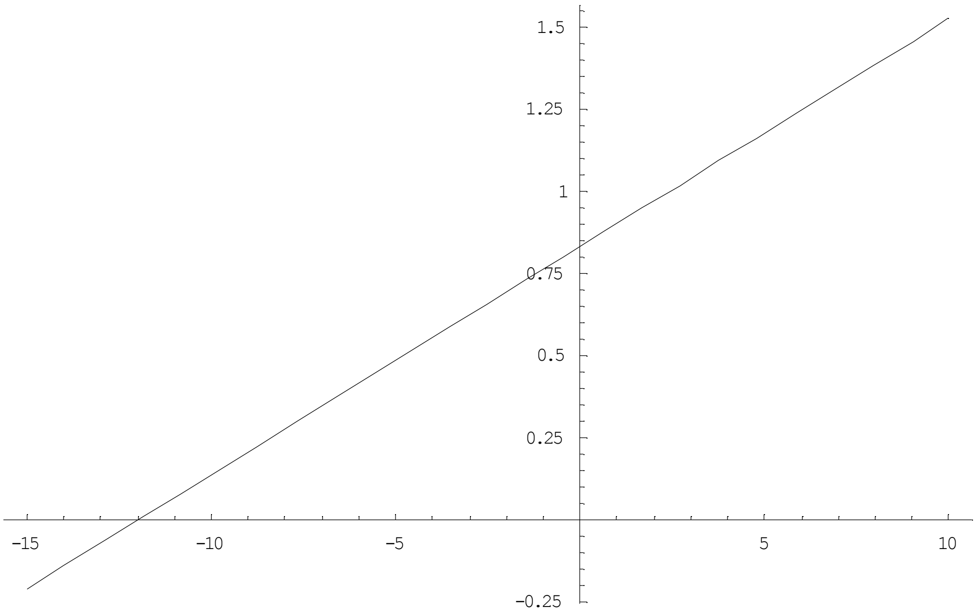
$$y = \frac{5}{72}x + \frac{5}{6}$$

$$y = \frac{5}{2}x - 5$$

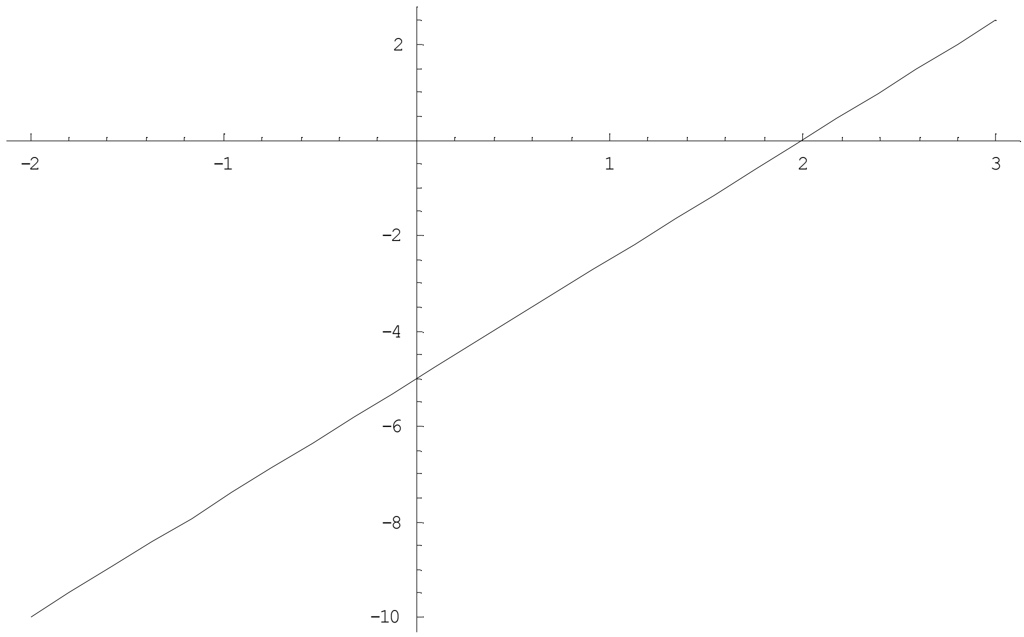
$$y = -\frac{5}{8}x + \frac{5}{2}$$

$$y = -\frac{5}{18}x + \frac{5}{3}$$

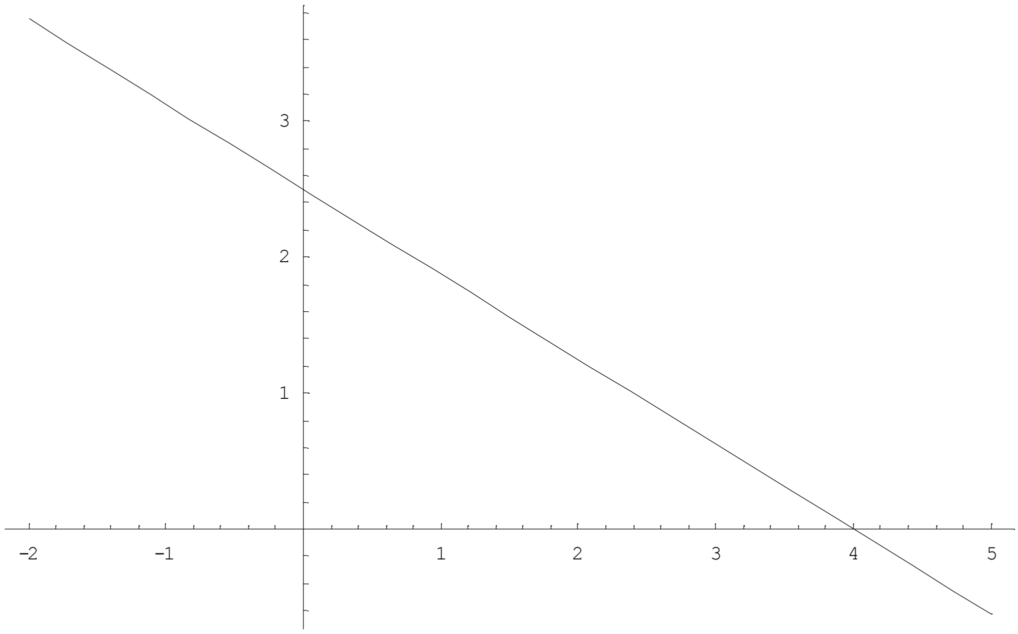
Let’s draw the plots:



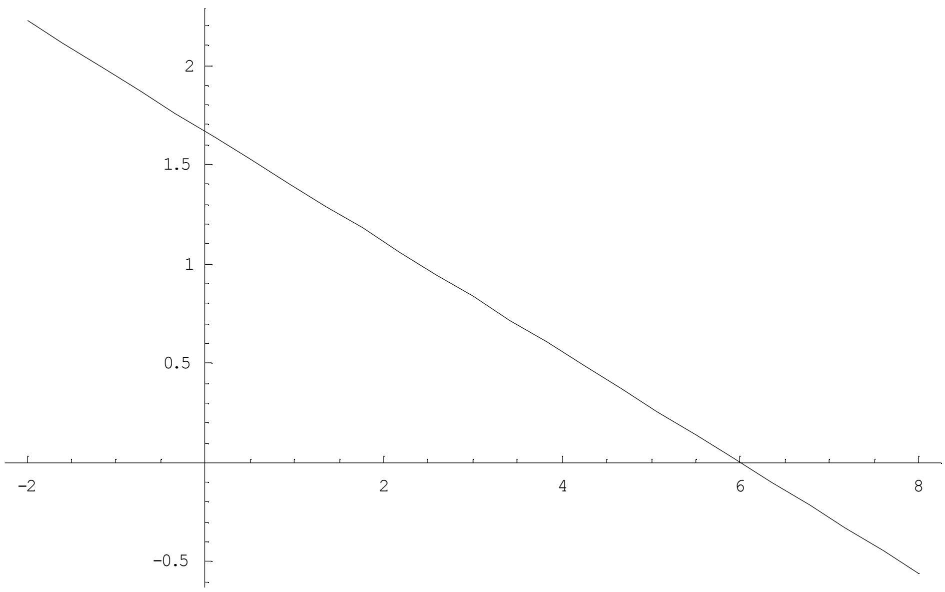
The plot of $y = \frac{5}{72}x + \frac{5}{6}$



The plot of $y = \frac{5}{2}x - 5$



The plot of $y=-\frac{5}{8}x+\frac{5}{2}$



The plot of $-\frac{5}{18}x+\frac{5}{3}$

ANSWER

$y=\frac{5}{72}x+\frac{5}{6}$

$y=\frac{5}{2}x-5$

$y=-\frac{5}{8}x+\frac{5}{2}$

$y=-\frac{5}{18}x+\frac{5}{3}$