Task:

Consider the equation $y \cdot u_x - x \cdot u_y = 0, (y \geq 0)$. Check for each of the following initial conditions whether the problem is solvable. If it is solvable, find a solution. If it is not, explain why:

(a) u(x,0) = x^2

(b) u(x,0) = x

(c) u(x,0) = x, $x > 0$

Solution:

$$y \cdot u_x - x \cdot u_y = 0, (y \geq 0)$$

$$\frac{dx}{y} = \frac{dy}{-x}$$

$$-xdx = ydy$$

$$ydy + xdx = 0$$

$$\varphi(x, y) = \frac{y^2}{2} + \frac{x^2}{2}$$

$$u(x, y) = c_1\left(\frac{y^2}{2} + \frac{x^2}{2}\right) = c_2(y^2 + x^2)$$

(a)

$$u(x, 0) = x^2$$

$$u(x, 0) = c_2(0^2 + x^2)$$

$$c_2 = 1$$

$$u(x, y) = y^2 + x^2, y > 0$$

(b)

$$u(x, 0) = x$$

$$u(x, 0) = c_2(0^2 + x^2) = c_2x^2$$

$$c_2 = \frac{1}{x}$$

$$u(x, y) = \frac{y^2}{x} + x, y \geq 0$$

(c)

$$u(x, y) = \frac{y^2}{x} + x, x > 0, y \geq 0$$

Answer:

(a) $u(x, y) = y^2 + x^2, y > 0$

(b) $u(x, y) = \frac{y^2}{x} + x, y \geq 0$

(c) $u(x, y) = \frac{y^2}{x} + x, x > 0, y \geq 0$