Question 1. *Decide whether the set of all real functions form a vector space.*

Solution. Recall that the set V of all functions $X \to \mathbb{R}$ has the natural addition and multiplication on a scalar from $\mathbb{R}$:

$$(f+g)(x) = f(x) + g(x),$$
$$(\alpha f)(x) = \alpha f(x).$$

Prove that V is a vector space over $\mathbb{R}$. Check the vector space axioms.

Associativity and commutativity of addition follow from the corresponding properties of addition in $\mathbb{R}$.

The function which is 0 for all $x \in \mathbb{R}$ is the zero of V.

The function $(-f)(x) = -f(x)$ is the inverse of f under addition.

Distributivity of scalar multiplication with respect to vector addition and distributivity of scalar multiplication with respect to field addition follow from the distributivity of multiplication with respect to addition in $\mathbb{R}$.

Compatibility of scalar multiplication with field multiplication follows from the associativity of multiplication in $\mathbb{R}$.

The identity 1 of $\mathbb{R}$ acts as the identity of scalar multiplication, because $(1 \cdot f)(x) = 1 \cdot f(x) = f(x)$. $\square$