Question 1. *Show that for every $a, b \in \mathbb{Q}$ if $a.b = a + b + ab$, then $(\mathbb{Q}, .)$ is a group.*

Solution. Obviously, the operation . is commutative. Prove the associativity of $(\mathbb{Q}, .)$:

$$a.(b.c) = a.(b + c + bc) = a + b + c + bc + ab + ac + abc,$$
$$(a.b).c = (a + b + ab).c = a + b + ab + c + ac + bc + abc,$$

so $a.(b.c) = (a.b).c$. Show that 0 is the identity of $(\mathbb{Q}, .)$. Indeed, for any $a \in \mathbb{Q}$ we have

$$a.0 = a + 0 + a \cdot 0 = a.$$

Furthermore, for arbitrary $a \in \mathbb{Q}$ find the inverse of a in $(\mathbb{Q}, .)$. Suppose

$$a.b = 0 \Leftrightarrow a + b + ab = 0 \Leftrightarrow a + b(a + 1) = 0.$$

Now we see that for $a = -1$ we get $a = 0$, which is a contradiction. So, $a = -1$ cannot be invertible in $(\mathbb{Q}, .)$. Thus, $(\mathbb{Q}, .)$ is not a group.

But if we throw out $a = -1$ and consider $(\mathbb{Q} \setminus \{-1\}, .)$, we shall obtain a group. Indeed, first of all prove that $(\mathbb{Q} \setminus \{-1\}, .)$ is closed under .: suppose $a.b = -1$ for some $a, b \neq -1$, i. e.

$$a + b + ab = -1 \Leftrightarrow a + ab = -(b + 1) \Leftrightarrow a(b + 1) = -(b + 1).$$

But if $b \neq -1$, then $b + 1 \neq 0$, so we can reduce and get $a = -1$, a contradiction. Now show that each $a \in \mathbb{Q} \setminus \{-1\}$ is invertible in $(\mathbb{Q} \setminus \{-1\}, .)$. As we have seen above:

$$a.b = 0 \Leftrightarrow a + b(a + 1) = 0 \Leftrightarrow b = -\frac{a}{a + 1} \in \mathbb{Q} \setminus \{-1\},$$

which is defined, since $a \neq -1$. $\qquad \square$