Let $u = 1 - x2 - x3$, $v = 1 - x - x4$, $R = Z[\zeta]$ where ζ is a primitive 5th root of unity. By Dirichlet's Unit Theorem, U(R) has rank 1. To shorten the proof, we shall assume the number-theoretic fact that $1 + \zeta$ is a fundamental unit in R, that is, U(R) = <1 + ζ > $\{\pm\zeta^i\}$.

(Of course, $1 + \zeta$ has infinite order. Its inverse is $-(\zeta + \zeta^3)$.) Now consider the projection $\varphi : ZG \to R$ defined by $\varphi(x) = \zeta$. We claim that φ : U(ZG) $\to$ U(R) is an injection. To see this, suppose $\varphi(\alpha_0 + \alpha_1 x + \cdots + \alpha_4 x^4) = 1$, where $\alpha_i \in Z$. This implies that $1 = \alpha_0 + \alpha_1\zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3 + \alpha_4(-1 - \zeta - \zeta^2 - \zeta^3) = (\alpha_0 - \alpha_4) + (\alpha_1 - \alpha_4)\zeta + (\alpha_2 - \alpha_4)\zeta^2 + (\alpha_3 - \alpha_4)\zeta^3$, so we have $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ (say), and $\alpha_0 = 1+\alpha$. On the other hand, the "augmentation" $\alpha_0 + \alpha_1 + \cdots + \alpha_4 = (1+\alpha) + 4\alpha = 1 + 5\alpha$ must be ±1, since $\sum \alpha_i x^i \in$ U(ZG). Therefore, we must have $\alpha = 0$, and $\sum \alpha_i x^i = 1$, which proves our claim.

We can now determine U(ZG) by computing φ(U(ZG)). Of course, $\varphi(\pm G) = \{\pm\zeta^i\}$. A simple calculation using the same ideas as above shows that $1 + \zeta \notin \varphi$(U(ZG)). On the other hand, $uv = 1$ (by direct calculation), and

$\varphi(xu) = \zeta(1 - \zeta^2 - \zeta^3) = \zeta - \zeta^3 + (1 + \zeta + \zeta^2 + \zeta^3) = 1 + 2\zeta + \zeta^2 = (1+\zeta)^2$.

This, together with $\varphi(\pm G) = \{\pm\zeta^i\}$, imply that [U($R$) : φ(U(ZG))] ≤ 2. Since $1 + \zeta \notin \varphi$(U(ZG)), equality must hold, and we must have φ(U(ZG)) = <(1 + ζ)>$\cdot$ $\{\pm\zeta^i\}$, and therefore

U(ZG) = <xu>$\times$ ($\pm G$) = <u> $\times$ ($\pm G$) ~ Z $\oplus$ Z2 $\oplus$ Z5.