Write $G = \cup_i x_i H$. First assume char $k \nmid [G : H]$. Since $kG = \oplus_i x_i kH$, any $\alpha \in$ rad kG can be written in the form $\Sigma x_i \alpha_i$ where $\alpha_i \in kH$. We claim that each $\alpha_i \in$ rad kH. Indeed, consider any simple kH-module W. Then, $kG \otimes_{kH} W$ is a semisimple kG-module, so $0 = \alpha \cdot kG \otimes_{kH} W \supseteq \Sigma x_i \alpha_i \otimes_{kH} W = \Sigma x_i \otimes \alpha_i W$. This implies that $\alpha_i W = 0$ for all i, so $\alpha_i \in$ rad kH. We have now shown that rad $kG \subseteq kG \cdot$ rad kH, and the reverse inclusion follows from $kH \cap$ rad kG = rad kH. (Note that the work above actually yields an explicit dimension equation: $\dim_k$ rad kG = $[G : H] \dim_k$ rad kH.)

For the converse, let us now assume that rad $kG = kG \cdot$ rad kH. Consider the group algebra $R = k[G/H]$ as a left kG-module (via the action of G on G/H). Since H acts as the identity on R, the augmentation ideal of kH acts as zero, and therefore so do rad kH and rad $kG = kG \cdot$ rad kH. We can then view R as a $kG/$rad kG-module. Since $kG/$rad kG is a semisimple ring, R is a semisimple $kG/$rad kG-module. Therefore, R is also a semisimple module over kG and over R, so R is itself a semisimple ring. Now we have that char $k \nmid [G : H]$.