Show that each of the following equations has a solution of the form

$$u(x,y) = e^{ax+by}.$$

Find the constants a,b for each example:

a) $u_x + 3u_y + u = 0.$

b) $u_{xx} + u_{yy} = 5e^{x-2y}.$

Solution:

a)

We have

$$u_x(x,y) = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(e^{ax+by}) = ae^{ax+by};$$

$$u_y(x,y) = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(e^{ax+by}) = be^{ax+by}.$$

Then

$$u_x + 3u_y + u = 0,$$
$$ae^{ax+by} + 3be^{ax+by} + e^{ax+by} = 0,$$
$$(a + 3b + 1)e^{ax+by} = 0.$$

So the function $u(x,y) = e^{ax+by}$ will be the solution of the equation $u_x + 3u_y + u = 0$ if

$$a + 3b + 1 = 0,$$
$$a = -3b - 1, \qquad b \in R.$$

b)

We have

$$u_{xx}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(ae^{ax+by}) = a^2e^{ax+by};$$

$$u_{yy}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(be^{ax+by}) = b^2e^{ax+by}.$$

Then

$$u_{xx} + u_{yy} = 5e^{x-2y},$$

$$a^2e^{ax+by} + b^2e^{ax+by} = 5e^{x-2y},$$
$$(a^2 + b^2)e^{ax+by} = 5e^{x-2y}.$$

So the function $u(x,y) = e^{ax+by}$ will be the solution of the equation $u_{xx} + u_{yy} = 5e^{x-2y}$ if

$$\begin{cases} a^2 + b^2 = 5, \\ ax + by = x - 2y. \end{cases}$$

So we have $a = 1,\ b = -2.$

Answer:

a) $a = -3b - 1,\ b \in R.$

b) $a = 1,\ b = -2.$